

# Robust Mean Field Linear-Quadratic-Gaussian Games with Unknown $L^2$ -Disturbance\*

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## Abstract

This paper considers a class of mean field linear-quadratic-Gaussian (LQG) games with model uncertainty. The drift term in the dynamics of the agents contains a common unknown function. We take a robust optimization approach where a representative agent in the limiting model views the drift uncertainty as an adversarial player. By including the mean field dynamics in an augmented state space, we solve two optimal control problems sequentially, which combined with consistent mean field approximations provides a solution to the robust game. A set of decentralized control strategies is derived by use of forward-backward stochastic differential equations (FBSDE) and shown to be a robust  $\varepsilon$ -Nash equilibrium.

## 1 Introduction

Mean field game theory provides an effective methodology for the analysis and strategy design in a large population of players which are individually insignificant but collectively have strong impact (see e.g. [24, 27, 28, 34]). A typical modeling analyzes a system of  $N$  players with mean field coupling in their dynamics or costs, or both. The linear-quadratic-Gaussian (LQG) framework is of particular interest since it allows an explicit solution procedure. Consider a large population of  $N$  agents. The dynamics of agent  $i$  are given by the stochastic differential equation (SDE)

$$dx_i(t) = (Ax_i(t) + Bu_i(t) + Gx^{(N)}(t))dt + DdW_i(t), \quad t \geq 0, \quad (1)$$

where  $x^{(N)} = (1/N) \sum_{i=1}^N x_i$  denotes the mean field coupling term. The cost of agent  $i$  is given by

$$\mathcal{J}_i(u_i, \dots, u_N) = \mathbb{E} \left[ \int_0^T (|x_i - \Gamma x^{(N)} - \eta|_Q^2 + u_i^T R u_i) dt + x_i^T(T) H x_i(T) \right], \quad (2)$$

where we denote  $|z|_Q = (z^T Q z)^{\frac{1}{2}}$  and the symmetric matrices  $Q \geq 0, H \geq 0$  and  $R > 0$ . The LQG modeling framework was first developed in [24, 27] to obtain a set of strategies  $(\hat{u}_1, \dots, \hat{u}_N)$  such that each  $\hat{u}_i$  only uses the local sample path information of  $x_i$  and some deterministic functions reflecting the collective behavior of the agents and such that  $(\hat{u}_1, \dots, \hat{u}_N)$  is an  $\varepsilon$ -Nash equilibrium. There has existed a substantial body of literature adopting the LQG framework [4, 7, 31, 35, 43, 47].

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For further literature, the reader is referred to [12, 13, 14, 20, 28, 32] for nonlinear diffusion based games and the associated SDE analysis, [11, 34] for study of the coupled system of Hamilton-Jacobi-Bellman (HJB) and Fokker-Planck equations, [6, 9, 26, 41, 42] for models containing a major player, [8, 18] for time consistent strategies in mean field games, [52] for mean field oscillator games, [50] for Markovian switching mean field games, [15] for application to Bertrand and Cournot equilibrium models, and [1] for a related solution notion called stationary equilibrium where players optimize assuming a steady-state long-run average for the empirical distribution of others' states. For an overview on mean field game theory, see [6, 10, 21].

Within the traditional research on games, there has existed a fair amount of literature on model uncertainty. For an  $N$  player static game with finite action spaces and an uncertain payoff matrix, a robust-optimization equilibrium is introduced in [2] where each player optimizes its worst case payoff with respect to the uncertain set. A similar method is applied to hierarchical static games [22]. Robustness has been addressed in dynamic games as well. A linear-quadratic (LQ) game with system parameter uncertainties is presented in [29], and the deviation from the Nash equilibrium is estimated for a set of nominal strategies. Robust Nash equilibria are analyzed in [49] for an LQ game with an unknown time-varying disturbance signal as an adversarial player. In the first case, a soft-constrained game is solved where the cost includes a quadratic penalty term for the disturbance. The second case introduces a hard constraint by specifying an  $L^2$  bound on the disturbance function. The work [30] deals with stochastic games where the payoff and state transition probabilities contain uncertainty. The solution is developed by letting each player solve a robust Markov decision problem to optimize its worst case cost while other players' strategies are fixed.

This paper aims to address model uncertainty in the mean field LQG game context. Specifically, we focus on drift uncertainty by adding to (1) a common unknown  $L^2$ -disturbance  $f$ . A practical motivation is that in many decision problems, a large number of agents can share a common uncertainty source fluctuating with time, and examples include taxation, subsidy, interest rates, and so on. A direct consequence of our modeling is that this disturbance has global influence on the population. To address robustness, each agent locally views the disturbance as an adversarial player, and for this purpose we incorporate into (2) an effort penalty term for the disturbance which in turn maximizes the resulting cost first. The agent minimizes subsequently. The framework of letting the disturbance maximize while its effort is penalized is called the soft-constraint approach [5, 19, 49]. It has the advantage of analytical tractability. When a hard constraint is considered, the robust mean field game is more difficult to tackle; see some preliminary analysis in [25]. Regarding robustness in mean field games, a related work is [46] where each agent is paired with its local disturbance as an adversarial player. The resulting solution is to replace the usual HJB equation by a Hamilton-Jacobi-Isaacs (HJI) equation in the solution.

To design the individual strategies it is necessary to build the dynamics of the mean field (i.e. state average of the agents) evolving under the disturbance. This technique shares its spirit with the state augmentation method in major player models [26, 41, 42]. The subsequent robust optimization problem, as a minimax control problem, leads to two optimal control problems with indefinite state weights [51]. They are different from the well known stochastic control problems with indefinite control weights [17, 37]. We will follow a convex optimization approach to solve the two control problems via variational analysis and forward-backward stochastic differential equations (FBSDE) [23, 40, 44]. Both the information structure and the solution procedure for our model are different from [46] where each player and its local disturbance have access to its state and so dynamic programming is applicable. Our main contributions are summarized as follows:

- We formulate a class of mean field LQG games where the players face a common uncertainty source, and introduce the robust optimization approach to solve two convex optimal control problems.

- Decentralized strategies are obtained for the robust mean field game via a set of FBSDE.
- The performance of the decentralized strategies for the  $N$  players is characterized as a robust  $\varepsilon$ -Nash equilibrium.

The rest of this paper is organized as follows. Section 2 introduces the mean field LQG game with a common disturbance and defines the worst case cost for a player. Section 3 studies the limiting robust optimization problem which leads to two optimal control problems solved sequentially by the disturbance and the representative player. The solution equation system of the mean field game is obtained in Section 4 based on consistent mean field approximations. A key error estimate of the mean field approximation is developed in Section 5. Section 6 characterizes the set of decentralized strategies as a robust  $\varepsilon$ -Nash equilibrium. An extension of the analysis to players with random initial states is presented in Section 7, and Section 8 concludes the paper.

## 2 Mean Field LQG Games with Drift Uncertainty

Consider a finite time horizon  $[0, T]$  for  $T > 0$ . Suppose that  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$  is a complete filtered probability space. Throughout this paper, we denote by  $\mathbb{R}^k$  the  $k$ -dimensional Euclidean space,  $\mathbb{R}^{n \times k}$  the set of all  $n \times k$  matrices. We use  $|\cdot|$  to denote the norm of a Euclidean space, or the Frobenius norm for matrices. For a vector or matrix  $M$ ,  $M^T$  denotes its transpose. Let  $L^2_{\mathcal{F}}(0, T; \mathbb{R}^k)$  denote the space of all  $\mathbb{R}^k$ -valued  $\mathcal{F}_t$ -progressively measurable processes  $x(\cdot)$  satisfying  $\mathbb{E} \int_0^T |x(t)|^2 dt < \infty$ ;  $C([0, T]; \mathbb{R}^k)$  (resp.,  $C^1([0, T]; \mathbb{R}^k)$ ) is the space of all  $\mathbb{R}^k$ -valued functions  $h(\cdot)$  defined on  $[0, T]$  which are continuous (resp., continuously differentiable);  $L^2(0, T; \mathbb{R}^k)$  is the space of all  $\mathbb{R}^k$ -valued measurable functions  $h(\cdot)$  on  $[0, T]$  satisfying  $\int_0^T |h(t)|^2 dt < \infty$ , and we denote the norm  $\|h\|_{L^2} = (\int_0^T |h(t)|^2 dt)^{1/2}$ . Throughout the paper, we use  $C$  (or  $C_1, C_2, \dots$ ) to denote a generic constant which does not depend on the population size  $N$  and may vary from place to place.

### 2.1 The game with a finite population

Consider  $N$  agents (or players) denoted by  $\mathcal{A}_i$ ,  $1 \leq i \leq N$ , respectively. The state  $x_i$  of  $\mathcal{A}_i$  is  $\mathbb{R}^n$ -valued and satisfies the linear SDE

$$dx_i(t) = (Ax_i(t) + Bu_i(t) + Gx^{(N)}(t) + f(t))dt + DdW_i(t), \quad 1 \leq i \leq N, \quad (3)$$

where  $x^{(N)} = (1/N) \sum_{j=1}^N x_j$ . The control  $u_i$  takes its value in  $\mathbb{R}^{n_1}$ . The  $\mathbb{R}^{n_2}$ -valued standard Brownian motions  $\{W_i(t), 1 \leq i \leq N\}$  are independent. The initial states  $\{x_i(0), 1 \leq i \leq N\}$  are deterministic and their empirical mean has the limit  $\lim_{N \rightarrow \infty} (1/N) \sum_{i=1}^N x_i(0) = m_0$ . We take  $\{\mathcal{F}_t\}_{0 \leq t \leq T}$  as the natural filtration generated by the  $Nn_2$ -dimensional Brownian motion  $(W_1(t), \dots, W_N(t))$ , and  $\mathcal{F} = \mathcal{F}_T$ . The admissible control set  $\mathcal{U}$  of  $\mathcal{A}_i$  is

$$\mathcal{U} := \{u_i(\cdot) : u_i \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})\}.$$

Denote  $u = (u_1, \dots, u_N)$  and  $u_{-i} = (u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_N)$ .

The function  $f \in L^2(0, T; \mathbb{R}^n)$  is an unknown disturbance to characterize the model uncertainty, and represents an influence from the common environment for decision-making. A natural motivation for considering deterministic disturbance is the following. Although each player  $\mathcal{A}_i$  regards the disturbance as adversarial, it *should not* be excessively pessimistic by assuming that the latter will use the sample path information of  $W_i$  to play against it, and instead only considers a deterministic  $f$ .

The cost functional of  $\mathcal{A}_i$  is

$$J_i(u_i, u_{-i}, f) = \mathbb{E} \left[ \int_0^T \left( |x_i - (\Gamma x^{(N)} + \eta)|_Q^2 + u_i^T R u_i - \frac{1}{\gamma} |f(t)|^2 \right) dt + x_i^T(T) H x_i(T) \right], \quad (4)$$

where the symmetric matrices  $Q \geq 0$ ,  $R > 0$ ,  $H \geq 0$  and the constant  $\gamma > 0$ . We assume uniform agents in the sense that they share the same parameter datum  $(A, B, G, D; \Gamma, \eta, Q, R, \gamma, H)$ . Also, to simplify the analysis, we consider constant parameters.

Due to the unknown function  $f$ ,  $\mathcal{A}_i$  cannot evaluate its cost even if all control policies  $(u_1, \dots, u_N)$  are known. To address this indeterminacy, we approach the game from a robust optimization point of view where each agent takes  $f$  as an adversarial player. Here a soft-constraint [5, 49, 19] for the disturbance is adopted in that the term  $-\frac{1}{\gamma}|f(t)|^2$  is included in (4) while  $f$  attempts to maximize. For given  $(u_i, u_{-i})$ , define the worst case cost of  $\mathcal{A}_i$  as

$$J_i^{\text{wo}}(u_i, u_{-i}) = \sup_{f \in L^2(0, T; \mathbb{R}^n)} J_i(u_i, u_{-i}, f).$$

A set of strategies  $(\hat{u}_1, \dots, \hat{u}_N)$  is a robust  $\varepsilon$ -Nash equilibrium for the  $N$  players if for  $\varepsilon \geq 0$ ,

$$J_i^{\text{wo}}(\hat{u}_i, \hat{u}_{-i}) - \varepsilon \leq \inf_{u_i \in \mathcal{U}} J_i^{\text{wo}}(u_i, \hat{u}_{-i}) \leq J_i^{\text{wo}}(\hat{u}_i, \hat{u}_{-i}). \quad (5)$$

Our central objective is to design decentralized strategies based on the above solution notion.

### 3 The Limiting Robust Optimization Problem

We start by making an appropriate approximation of the coupling term  $x^{(N)}$ . Adding up the  $N$  equations in (3) and normalizing by  $1/N$ , we obtain

$$dx^{(N)} = [(A + G)x^{(N)} + Bu^{(N)} + f]dt + D(1/N) \sum_{j=1}^N dW_j,$$

where  $u^{(N)} = (1/N) \sum_{j=1}^N u_j$ . Intuitively, from the point of view of  $\mathcal{A}_i$ ,  $u^{(N)}$  may be approximated by a deterministic function  $\bar{u}$ . Moreover, when  $N \rightarrow \infty$ ,  $(1/N) \sum_{j=1}^N dW_j$  vanishes due to the law of large numbers. In turn, a deterministic function  $m$  can be used to approximate  $x^{(N)}$ . The above reasoning suggests to introduce the limiting ordinary differential equation (ODE)

$$\dot{m} = (A + G)m + B\bar{u} + f, \quad m(0) = m_0. \quad (6)$$

#### 3.1 The limiting model of the mean field game

Consider the optimization problem of a representative agent  $\mathcal{A}_i$ :

$$\begin{cases} dx_i = (Ax_i + Bu_i + Gm_i + f)dt + DdW_i, \\ \dot{m}_i = (A + G)m_i + B\bar{u} + f, \end{cases} \quad (7)$$

where the second equation is motivated from (6) and  $m_i(0) = m_0$ . For the limiting model (7),  $(W_i, x_i(0))$  is the same as in (3). We reuse  $(x_i, \mathcal{A}_i)$  to denote the state and the corresponding agent. This shall cause no risk of confusion. Since  $f$  will be determined as its worst case form depending on  $x_i(0)$ ,  $m_i$  is associated with the agent index  $i$  so that it is ready as an appropriate notation for the subsequent closed-loop dynamics. The cost functional is given by

$$\bar{J}_i(u_i, f) = \mathbb{E} \int_0^T \left\{ |x_i - (\Gamma m_i + \eta)|_Q^2 + u_i^T R u_i - \frac{1}{\gamma} |f(t)|^2 \right\} dt + \mathbb{E} x_i^T(T) H x_i(T).$$

We aim to find a solution pair  $(\hat{f}, \hat{u}_i)$  such that

$$\bar{J}_i(\hat{u}_i, \hat{f}) = \min_{u_i \in \mathcal{U}} \max_{f \in L^2(0, T; \mathbb{R}^n)} \bar{J}_i(u_i, f). \quad (8)$$

Finally, we need a consistency condition, i.e.,  $\frac{1}{N} \sum_{i=1}^N \hat{u}_i$  converges to  $\bar{u}$  in some sense (this will be made precise in Section 4) and we look for  $\bar{u} \in C([0, T]; \mathbb{R}^{n_1})$ ; the feasibility of doing so will be clear from our solution procedure. The next part of our plan is to show that such strategies have the property in (5) when applied in the game of  $N$  agents. In the following, we solve the optimization problem (8) in two steps.

### 3.2 The control problem with respect to the disturbance

Let  $u_i \in \mathcal{U}$  and  $\bar{u} \in C([0, T]; \mathbb{R}^{n_1})$  be fixed. The optimal control problem is

$$(\mathbf{P1}) \quad \text{maximize}_{f \in L^2(0, T; \mathbb{R}^n)} \bar{J}_i(u_i, f). \quad (9)$$

Clearly (P1) is equivalent to the following problem

$$(\mathbf{P1a}) \quad \text{minimize}_{f \in L^2(0, T; \mathbb{R}^n)} \bar{J}_i'(u_i, f) = \mathbb{E} \int_0^T \left\{ -|x_i - (\Gamma m_i + \eta)|_Q^2 + \frac{1}{\gamma} |f(t)|^2 \right\} dt \\ - \mathbb{E} x_i^T(T) H x_i(T).$$

(P1a) is an optimal control problem with negative semi-definite state weights. We are interested in the situation where (P1a) is a strictly convex problem with a coercivity property. This ensures that the worst case disturbance is uniquely determined by  $\mathcal{A}_i$ . The procedure below to identify conditions for ensuring convexity is similar to [37].

To study the convexity of  $\bar{J}_i'$  in  $f$ , we construct a simpler auxiliary optimal control problem. Denote

$$\hat{Q} = (I - \Gamma)^T Q (I - \Gamma).$$

Consider the dynamics

$$\dot{z} = (A + G)z + g, \quad z(0) = 0, \quad (10)$$

where  $g \in L^2(0, T; \mathbb{R}^n)$ . The optimal control problem is

$$(\mathbf{P1b}) \quad \text{minimize} \quad \bar{J}_i''(g) = \int_0^T \left\{ -z^T \hat{Q} z + \frac{1}{\gamma} |g(t)|^2 \right\} dt - z^T(T) H z(T).$$

For any  $s \in \mathbb{R}$ , we have  $\bar{J}_i''(sg) = s^2 \bar{J}_i''(g)$ , and so view  $\bar{J}_i''$  as a quadratic functional of  $g$ .

**Definition 1** Let  $F(g)$  be a real-valued functional of  $g \in L^2(0, T; \mathbb{R}^n)$ . If  $F(g) \geq 0$  for all  $g$ ,  $F$  is said to be positive semi-definite. If furthermore,  $F(g) > 0$  for all  $g \neq 0$ ,  $F$  is said to be positive definite.

**Lemma 2**  $\bar{J}_i'(u_i, f)$  is convex (resp., strictly convex) in  $f$  if and only if  $\bar{J}_i''(g)$  is positive semi-definite (resp., positive definite).

*Proof.* Let  $(x_i, m_i)$  and  $(x'_i, m'_i)$  be the state processes of (7) corresponding to  $(u_i, f)$  and  $(u_i, f')$ , respectively. Take any  $\lambda_1 \in [0, 1]$  and denote  $\lambda_2 = 1 - \lambda_1$ . Then

$$\lambda_1 \bar{J}_i'(u_i, f) + \lambda_2 \bar{J}_i'(u_i, f') - \bar{J}_i'(u_i, \lambda_1 f + \lambda_2 f') \\ = \lambda_1 \lambda_2 \mathbb{E} \int_0^T \left\{ |x_i - x'_i - \Gamma(m_i - m'_i)|_Q^2 + \frac{1}{\gamma} |f(t) - f'(t)|^2 \right\} dt - \lambda_1 \lambda_2 \mathbb{E} |x_i(T) - x'_i(T)|_H^2.$$

Denote  $g = f - f'$ , and  $z = x_i - x'_i$ . Therefore,  $z$  is deterministic and satisfies (10). In addition,  $m_i - m'_i = z$  for  $t \in [0, T]$ . Hence

$$\lambda_1 \bar{J}'_i(u_i, f) + \lambda_2 \bar{J}'_i(u_i, f') - \bar{J}'_i(u_i, \lambda_1 f + \lambda_2 f') = \lambda_1 \lambda_2 \bar{J}''_i(g)$$

and the lemma follows.  $\square$

For our further existence analysis, we need to ensure  $\bar{J}'_i(u_i, f)$  to be both strictly convex and coercive in  $f$ . For this purpose, we introduce the following assumption.

**(H1)** There exists a small  $\epsilon_0 > 0$  such that  $\bar{J}''_i(g) - \epsilon_0 \|g\|_{L^2}^2$  is positive semi-definite.

Note that (H1) is completely determined by the parameters  $(\hat{Q}, \gamma, \epsilon_0, H, T)$ , and does not depend on  $u_i$ . Concerning (H1), we have the following result.

**Proposition 3** *The following statements are equivalent:*

- (i) (H1) holds true on  $[0, T]$ .
- (ii) The Riccati equation

$$\dot{P} + (A + G)^T P + P(A + G) - \gamma P^2 - \hat{Q} = 0, \quad P(T) = -H \quad (11)$$

has a unique solution on  $[0, T]$ .

- (iii) For any  $t \in [0, T]$ ,

$$\det\{[(0, I)e^{At}(0, I)^T]\} > 0,$$

where  $\mathcal{A} = \begin{pmatrix} A + G + \gamma H & -\gamma I \\ \check{Q} & -(A + G + \gamma H)^T \end{pmatrix}$  and  $\check{Q} = \gamma H^2 + \hat{Q} + (A + G)^T H + H(A + G)$ .

*Proof.* In fact, (H1) is the uniform convexity condition proposed in [45], and the equivalence between (i) and (ii) is a corollary of Theorem 4.6 of [45]. Moreover, (iii)  $\implies$  (ii) is given in Theorem 4.3 of [40]. On the other hand, (ii)  $\implies$  (iii) is implied by Theorems 2.7 and 2.9 of [54].  $\square$

For illustration of condition (ii), we give the following example.

**Example 4** Consider system (3)-(4) with parameters  $A = 0.5$ ,  $B = 1$ ,  $G = 0.25$ ,  $Q = 1$ ,  $\Gamma = 0.8$ ,  $R = 1.5$ ,  $H = 0$ ,  $\gamma = 1$ . Denote  $\hat{A} = A + G$ . We solve (11) to obtain

$$P(t) = \frac{-\hat{Q}(e^{\alpha(t-T)} - e^{-\alpha(t-T)})}{\lambda_2 e^{\alpha(t-T)} - \lambda_1 e^{-\alpha(t-T)}}, \quad (12)$$

where

$$\begin{aligned} \lambda_1 &= -\hat{A} + \sqrt{\hat{A}^2 - \gamma \hat{Q}} = -0.027158, & \lambda_2 &= -\hat{A} - \sqrt{\hat{A}^2 - \gamma \hat{Q}} = -1.472842, \\ \alpha &= \sqrt{\hat{A}^2 - \gamma \hat{Q}} = 0.722842. \end{aligned}$$

If  $0 < T < T_{\max} = \frac{1}{2\alpha} \log(\lambda_2/\lambda_1) = 2.752198$ ,  $P(t)$  given by (12) is well defined on  $[0, T]$ . By the local Lipschitz continuity property of the vector field in (11),  $P(t)$  is the unique solution.

Note that (11) is not a standard Riccati equation since the state weight matrix  $-\hat{Q}$  is not positive semi-definite. In general, the solvability of (11) cannot be ensured on an arbitrary time horizon. Condition (iii) enables us to determine the solvability of (11) on a given time horizon. Note that condition (iii) is equivalent to  $\det\{[(0, I)e^{At}(0, I)^T]\} \neq 0, \forall t \in [0, T]$  by noting  $\det\{[(0, I)e^{At}(0, I)^T]\}_{t=0} = 1$ . Condition (iii) is more checkable as illustrated by the following example.

**Example 5** Consider system (3)-(4) with parameters  $A = -0.5$ ,  $G = 0.25$ ,  $Q = 1$ ,  $\Gamma = 0.8$ ,  $H = 0$ ,  $\gamma = 1$ . We obtain  $\mathcal{A} = \begin{pmatrix} -0.25 & -1 \\ 0.04 & 0.25 \end{pmatrix}$ ,  $e^{\mathcal{A}t} = \begin{pmatrix} -\frac{1}{3}e^{\frac{3}{20}t} + \frac{4}{3}e^{-\frac{3}{20}t} & -\frac{10}{3}e^{\frac{3}{20}t} + \frac{10}{3}e^{-\frac{3}{20}t} \\ \frac{2}{15}e^{\frac{3}{20}t} - \frac{2}{15}e^{-\frac{3}{20}t} & \frac{4}{3}e^{\frac{3}{20}t} + \frac{1}{3}e^{-\frac{3}{20}t} \end{pmatrix}$ , and

$$\det\{[(0,1)e^{\mathcal{A}t}(0,1)^T]\} = \frac{4}{3}e^{\frac{3}{20}t} + \frac{1}{3}e^{-\frac{3}{20}t} > 0, \quad \forall t \geq 0. \quad (13)$$

Thus for any  $T > 0$ , (11) admits a unique solution on  $[0, T]$ . Therefore, (H1) holds true on  $[0, T]$ .

**Lemma 6** Assume (H1). Then  $\bar{J}'_i(u_i, f)$  is strictly convex in  $f$ . Moreover,  $\bar{J}'_i(u_i, f)$  is coercive in  $f$  and, in particular, there exists a constant  $C_{u_i, x_i(0)}$  depending on  $(u_i, x_i(0))$  such that

$$\bar{J}'_i(u_i, f) \geq \frac{\epsilon_0}{2}\|f\|_{L^2}^2 - C_{u_i, x_i(0)}.$$

*Proof.* Since  $\bar{J}''_i(g) - \epsilon_0\|g\|_{L^2}^2$  is positive semi-definite by (H1),  $\bar{J}''_i(g)$  is positive definite. By Lemma 2,  $\bar{J}'_i(u_i, f)$  is strictly convex in  $f$ . Following the method in proving Lemma 2, we can further show that  $\chi(f) := \bar{J}'_i(u_i, f) - \epsilon_0\|f\|_{L^2}^2$  is convex in  $f$ . By (7) and direct estimates, we can show

$$\sup_{\|f\|_{L^2} \leq 1} |\chi(f)| \leq C_{0, u_i, x_i(0)},$$

where the constant  $C_{0, u_i, x_i(0)}$  depends on  $(u_i, x_i(0))$ . Now consider  $f$  with  $\|f\|_{L^2} \geq 1$ . Define  $f_1 = \frac{f}{\|f\|_{L^2}}$ . The convexity of  $\chi(f)$  implies

$$\chi(f_1) \leq \frac{1}{\|f\|_{L^2}}\chi(f) + \frac{\|f\|_{L^2} - 1}{\|f\|_{L^2}}\chi(0) \leq \frac{1}{\|f\|_{L^2}}\chi(f) + C_{0, u_i, x_i(0)}. \quad (14)$$

Consequently, for  $\|f\|_{L^2} \geq 1$ , (14) gives

$$\chi(f) \geq -2C_{0, u_i, x_i(0)}\|f\|_{L^2}.$$

Hence for any  $f$ ,  $\chi(f) \geq -C_{0, u_i, x_i(0)}(2\|f\|_{L^2} + 1)$ . It follows that

$$\begin{aligned} \bar{J}'_i(u_i, f) &= \chi(f) + \epsilon_0\|f\|_{L^2}^2 \\ &\geq \epsilon_0\|f\|_{L^2}^2 - C_{0, u_i, x_i(0)}(2\|f\|_{L^2} + 1) \\ &\geq \frac{\epsilon_0}{2}\|f\|_{L^2}^2 - C_{u_i, x_i(0)} \end{aligned}$$

for some constant  $C_{u_i, x_i(0)}$ . □

**Theorem 7** Suppose that (H1) holds and let  $u_i \in \mathcal{U}$  and  $\bar{u}$  be fixed. Then

- (i)  $\bar{J}'_i(u_i, f)$  has a unique minimizer  $\hat{f}$ , or equivalently,  $\bar{J}_i(u_i, f)$  has a unique maximizer  $\hat{f}$ ;
- (ii) there exists a unique solution  $(x_i, m_i, p_i) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times L^2(0, T; \mathbb{R}^{2n})$  to the equation system

$$\begin{cases} dx_i = (Ax_i + Bu_i + Gm_i + \gamma p_i)dt + DdW_i, \\ \dot{m}_i = (A + G)m_i + B\bar{u} + \gamma p_i, \\ \dot{p}_i = -(A + G)^T p_i - (I - \Gamma)^T Q[\mathbb{E}x_i - (\Gamma m_i + \eta)], \end{cases} \quad (15)$$

where  $m_i(0) = m_0$  and  $p_i(T) = H\mathbb{E}x_i(T)$ , and furthermore  $\hat{f} = \gamma p_i$ .

*Proof.* (i) By Lemma 2,  $\bar{J}'_i$  is strictly convex and coercive. In addition,  $\bar{J}'_i$  is continuous in  $f$ . Hence there exists a unique  $\hat{f}$  such that  $\bar{J}'_i(u_i, \hat{f}) = \inf_f \bar{J}'_i(u_i, f)$  [33, Chap. 7], [39].

(ii) We start by establishing existence. Let the optimal state-control pair be denoted by  $(x_i, m_i, \hat{f})$ , which is uniquely determined. We have the relation

$$dx_i = (Ax_i + Bu_i + Gm_i + \gamma\hat{f})dt + DdW_i, \quad (16)$$

$$\dot{m}_i = (A + G)m_i + B\bar{u} + \gamma\hat{f}, \quad (17)$$

where  $m_i(0) = m_0$ . By using  $(x_i, m_i)$ , we obtain a unique solution  $p_i$  from

$$\dot{p}_i = -(A + G)^T p_i - (I - \Gamma)^T Q[\mathbb{E}x_i - (\Gamma m_i + \eta)], \quad (18)$$

where  $p_i(T) = H\mathbb{E}x_i(T)$ .

Now we consider another control  $f = \hat{f} + \tilde{f} \in L^2(0, T; \mathbb{R}^n)$  in place of  $\hat{f}$ . Let  $\tilde{x}_i$  and  $\tilde{m}_i$  be the first variations of  $x_i$  and  $m_i$ , respectively, which result from the variation  $\tilde{f}$  for  $\hat{f}$ . Then we have  $\tilde{x}_i = \tilde{m}_i$  for all  $t \in [0, T]$  and

$$\frac{d\tilde{x}_i}{dt} = (A + G)\tilde{x}_i + \tilde{f}, \quad \tilde{x}_i(0) = 0.$$

Since  $\bar{J}'_i$  has a minimum at  $(x_i, m_i, \hat{f})$ , the first variation of the cost satisfies

$$0 = \frac{\delta \bar{J}'_i}{2} = \mathbb{E} \int_0^T \left\{ -[x_i - (\Gamma m_i + \eta)]^T Q(I - \Gamma)\tilde{x}_i + \frac{1}{\gamma} \tilde{f}^T \tilde{f} \right\} dt - \mathbb{E}x_i^T(T)H\tilde{x}_i(T). \quad (19)$$

On the other hand,

$$\begin{aligned} \frac{d}{dt}(p_i^T \tilde{x}_i) &= \tilde{x}_i^T \dot{p}_i + p_i^T \frac{d\tilde{x}_i}{dt} \\ &= -[\mathbb{E}x_i - (\Gamma m_i + \eta)]^T Q(I - \Gamma)\tilde{x}_i + p_i^T \tilde{f}. \end{aligned} \quad (20)$$

Integrating both sides of (20) and invoking (19), we obtain

$$p_i^T(T)\tilde{x}_i(T) = \int_0^T \left( p_i^T \tilde{f} - \frac{1}{\gamma} \tilde{f}^T \tilde{f} \right) dt + \mathbb{E}x_i^T(T)H\tilde{x}_i(T). \quad (21)$$

Recalling  $p_i(T) = H\mathbb{E}x_i(T)$ , since  $\tilde{f}$  is arbitrary, it follows from (21) that

$$\hat{f} = \gamma p_i$$

for a.e.  $t \in [0, T]$ . Therefore,  $(x_i, m_i, p_i)$  determined by (16)-(18) is a solution to (15).

We proceed to show uniqueness. Suppose that  $(x'_i, m'_i, p'_i)$  is another solution of (15). Set the control  $f' = \gamma p'_i$ . It is straightforward to show that the first variation of  $\bar{J}'_i$  at the state control pair  $(x'_i, m'_i, f')$  is zero. Since  $\bar{J}'_i$  is strictly convex, this implies that  $(x'_i, m'_i, f')$  is the unique optimal state-control pair and so coincides with  $(x_i, m_i, \hat{f})$  where  $(x_i, m_i)$  is the optimal state process determined from (16)-(18). This further implies  $p'_i = p_i$ . So uniqueness follows. The last part of (ii) is now obvious.  $\square$



### 3.3 The control problem of player $\mathcal{A}_i$

Assume that (H1) holds. This will ensure that all the equation systems in this section have a well defined solution. The dynamics are given by

$$\begin{cases} dx_i = (Ax_i + Bu_i + Gm_i + \gamma p_i)dt + DdW_i, \\ \dot{m}_i = (A + G)m_i + B\bar{u} + \gamma p_i, \\ \dot{p}_i = -(A + G)^T p_i - (I - \Gamma)^T Q[\mathbb{E}x_i - (\Gamma m_i + \eta)], \end{cases} \quad (22)$$

where  $m_i(0) = m_0$  and  $p_i(T) = H\mathbb{E}x_i(T)$ . The optimal control problem is

$$\begin{aligned} (\mathbf{P2}) \quad \text{minimize}_{u_i \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})} \bar{J}_i(u_i, \hat{f}_{u_i}) &= \mathbb{E} \int_0^T \{ |x_i - (\Gamma m_i + \eta)|_Q^2 + u_i^T R u_i - \gamma |p_i(t)|^2 \} dt \\ &+ \mathbb{E} x_i^T(T) H x_i(T). \end{aligned}$$

Here we have taken  $\hat{f}_{u_i} = \gamma p_i$  which depends on  $u_i$ . We may simply write  $\bar{J}_i(u_i)$ . This is again a linear quadratic optimal control problem with indefinite weight for the state vector  $(x_i, m_i, p_i)$ . Note that a perturbation in  $u_i$  will cause a change of the mean term  $\mathbb{E}x_i$ . So this is essentially a mean field type optimal control problem; see related work [3, 53].

We continue to identify conditions under which (P2) is strictly convex and coercive. These conditions will be characterized by using an auxiliary control problem with dynamics

$$\begin{cases} \dot{z}_i = Az_i + B\nu_i + Gz + \gamma q, \\ \dot{z} = (A + G)z + \gamma q, \\ \dot{q} = -(A + G)^T q - (I - \Gamma)^T Q(z_i - \Gamma z), \end{cases} \quad (23)$$

where  $z_i(0) = z(0) = 0$  and  $q(T) = Hz_i(T)$ . The control  $\nu_i \in L^2(0, T; \mathbb{R}^{n_1})$ . The optimal control problem is

$$(\mathbf{P2a}) \quad \text{minimize} \quad \bar{J}_i^a(\nu_i) = \int_0^T \{ |z_i - \Gamma z|_Q^2 + \nu_i^T R \nu_i - \gamma |q(t)|^2 \} dt + |z_i(T)|_H^2. \quad (24)$$

We may view this as a deterministic optimal control problem with two point boundary value conditions for the state trajectory. We say  $\bar{J}_i^a$  is positive semi-definite if  $\bar{J}_i^a(\nu_i) \geq 0$  for all  $\nu_i$ ; if furthermore,  $\bar{J}_i^a(\nu_i) > 0$  whenever  $\nu_i \neq 0$ , we say  $\bar{J}_i^a$  is positive definite. In order to have a well defined optimal control problem, we need to show that (23) has a unique solution.

**Lemma 8** Assume (H1). For each  $\nu_i$ , there exists a unique solution  $(z_i, z, q) \in C^1([0, T]; \mathbb{R}^{3n})$  to (23).

*Proof.* Indeed, by taking  $u_i = 0$  and  $u_i = \nu_i \in L^2(0, T; \mathbb{R}^{n_1})$  in (22), we obtain two solutions  $(x_i^0, m_i^0, p_i^0)$  and  $(x_i^{\nu_i}, m_i^{\nu_i}, p_i^{\nu_i})$ , respectively. It is easy to show that  $(z_i, z, q) := (x_i^{\nu_i} - x_i^0, m_i^{\nu_i} - m_i^0, p_i^{\nu_i} - p_i^0)$  is a solution of (23) by observing that  $x_i^{\nu_i} - x_i^0$  is deterministic.

If there exist two different solutions to (23) for some  $\nu_i$ , then we can construct two different solutions to (22) for a given  $u_i$ , which is a contradiction to Theorem 7.  $\square$

**Lemma 9**  $\bar{J}_i(u_i)$  is convex (resp., strictly convex) in  $u_i \in \mathcal{U}$  if and only if  $\bar{J}_i^a(\nu_i)$  is positive semi-definite (resp., positive definite).

*Proof.* See appendix A.  $\square$

We introduce the following assumption.

**(H2)** There exists a small constant  $\delta_0 > 0$  such that  $\bar{J}_i^a(\nu_i) - \delta_0 \|\nu_i\|^2 \geq 0$  for all  $\nu_i \in L^2(0, T; \mathbb{R}^{n_1})$ .

### 3.4 Representation of the quadratic functional

We intend to find an expression of  $\bar{J}_i^a(\nu_i)$  so that (H2) can be characterized in a more explicit form. A change of coordinates will make the computation more convenient. Define  $\tilde{z} = z_i - z$ . Then (23) becomes

$$\begin{cases} \dot{\tilde{z}} = A\tilde{z} + B\nu_i, \\ \dot{z} = (A + G)z + \gamma q, \\ \dot{q} = -\hat{Q}z - (A + G)^T q - (I - \Gamma)^T Q\tilde{z}, \end{cases} \quad (25)$$

where  $\tilde{z}(0) = z(0) = 0$  and  $q(T) = H(\tilde{z}(T) + z(T))$ .

Define the Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A + G & \gamma I \\ -\hat{Q} & -(A + G)^T \end{bmatrix}$$

and the matrix ODE  $\dot{\Phi}(t) = \mathcal{H}\Phi(t)$  where  $\Phi(0) = I$ . Denote the partition

$$\Phi(t) = \begin{bmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \Phi_{21}(t) & \Phi_{22}(t) \end{bmatrix},$$

where each submatrix  $\Phi_{ij}$  is an  $n \times n$  matrix function.

We have

$$\tilde{z}(t) = \int_0^t e^{A(t-\tau)} B\nu_i(\tau) d\tau. \quad (26)$$

By solving  $(z, q)$  in (25), we obtain

$$\begin{aligned} z(t) &= \Phi_{12}(t)q(0) - \int_0^t \Phi_{12}(t-s)(I - \Gamma)^T Q\tilde{z}(s) ds, \\ q(t) &= \Phi_{22}(t)q(0) - \int_0^t \Phi_{22}(t-s)(I - \Gamma)^T Q\tilde{z}(s) ds, \end{aligned}$$

where  $q(0)$  is to be determined. At the terminal time,

$$z(T) = \Phi_{12}(T)q(0) - \int_0^T \Phi_{12}(T-s)(I - \Gamma)^T Q\tilde{z}(s) ds$$

and

$$\begin{aligned} q(T) &= \Phi_{22}(T)q(0) - \int_0^T \Phi_{22}(T-s)(I - \Gamma)^T Q\tilde{z}(s) ds \\ &= H\tilde{z}(T) + H\Phi_{12}(T)q(0) - H \int_0^T \Phi_{12}(T-s)(I - \Gamma)^T Q\tilde{z}(s) ds, \end{aligned}$$

where the second equality is due to the terminal condition of  $q$ . It follows that

$$[\Phi_{22}(T) - H\Phi_{12}(T)]q(0) = H\tilde{z}(T) + \int_0^T [\Phi_{22}(T-s) - H\Phi_{12}(T-s)](I - \Gamma)^T Q\tilde{z}(s) ds. \quad (27)$$

**Proposition 10** *If (H1) holds,  $\Phi_{22}(T) - H\Phi_{12}(T)$  is nonsingular.*

*Proof.* Under (H1), (25) has a unique solution by Lemma 8, and accordingly,  $q(0)$  is uniquely determined. If  $\Phi_{22}(T) - H\Phi_{12}(T)$  is singular, we may find two different solutions of  $q(0)$  from (27) which further give two different solutions to (25), leading to a contradiction. Hence,  $\Phi_{22} - H\Phi_{12}(T)$  is nonsingular.  $\square$

By solving  $q(0)$  in (27) and further eliminating  $\tilde{z}$ , we write  $z$  and  $q$  as integrals depending on  $\nu_i$ . Define the linear operator

$$[\mathcal{L}(\nu_i)](t) = \begin{bmatrix} \tilde{z}(t) \\ z(t) \\ q(t) \end{bmatrix}.$$

By standard estimates we can show that  $\mathcal{L}$  is a linear and bounded operator from  $L^2(0, T; \mathbb{R}^{n_1})$  to  $L^2(0, T; \mathbb{R}^{3n})$ . Let  $\mathcal{L}^*$  be its adjoint operator from  $L^2(0, T; \mathbb{R}^{3n})$  to  $L^2(0, T; \mathbb{R}^{n_1})$ . Define the operator

$$\mathcal{L}_T \nu_i = \tilde{z}(T) + z(T).$$

It can be shown that  $\mathcal{L}_T$  is a linear and bounded operator from  $L^2(0, T; \mathbb{R}^{n_1})$  to  $\mathbb{R}^n$ . Let  $\mathcal{L}_T^*$  be its adjoint operator. Now  $\bar{J}_i^a$  may be represented in terms of the inner product on  $L^2(0, T; \mathbb{R}^{n_1})$ :

$$\bar{J}_i^a(\nu_i) = \langle \Theta \nu_i, \nu_i \rangle + \langle R \nu_i, \nu_i \rangle + \langle \Theta_T \nu_i, \nu_i \rangle, \quad (28)$$

where

$$\Theta \nu_i = \mathcal{L}^* \begin{bmatrix} Q & Q(I - \Gamma) & 0 \\ (I - \Gamma)^T Q & \widehat{Q} & 0 \\ 0 & 0 & -\gamma I \end{bmatrix} \mathcal{L} \nu_i, \quad \Theta_T \nu_i = \mathcal{L}_T^* H \mathcal{L}_T \nu_i.$$

### Proposition 11

- (i)  $\bar{J}_i(u_i)$  is convex in  $u_i \in \mathcal{U}$  if and only if  $\langle (\Theta + \Theta_T + R) \nu_i, \nu_i \rangle \geq 0$  for all  $\nu_i \in L^2(0, T; \mathbb{R}^{n_1})$ .
- (ii) (H2) holds if and only if there exists  $\delta_0 > 0$  such that  $\langle (\Theta + \Theta_T + R) \nu_i, \nu_i \rangle \geq \delta_0 \|\nu_i\|_{L^2}^2$  for all  $\nu_i \in L^2(0, T; \mathbb{R}^{n_1})$ .

*Proof.* (i) follows from Lemma 9 and the representation (28). (ii) follows from (28).  $\square$

The criterion in part (ii) of Proposition 11 still involves the operators  $\Theta$  and  $\Theta_T$  on an infinite dimensional space. Here we give a sufficient condition to endure (H2) based on some more computable parameters. It is clear that  $\langle (\Theta + \Theta_T + R) \nu_i, \nu_i \rangle \geq \int_0^T (|\nu_i(t)|_R^2 - \gamma |q(t)|^2) dt$ . For simplicity, we only consider the case  $H = 0$ , and simple computations lead to

$$\begin{aligned} q(t) &= \Phi_{22}(t) \Phi_{22}^{-1}(T) \int_0^T \Phi_{22}(T-s) (I - \Gamma)^T Q \int_0^s e^{A(s-\tau)} B \nu_i(\tau) d\tau ds \\ &\quad - \int_0^t \Phi_{22}(t-s) (I - \Gamma)^T Q \int_0^s e^{A(s-\tau)} B \nu_i(\tau) d\tau ds =: q_1(t) - q_2(t). \end{aligned}$$

Denote  $b_1 = \sup_{0 \leq t \leq T} |\Phi_{22}(t)|$ ,  $b_2 = \sup_{0 \leq t \leq T} |\Phi_{22}(t) \Phi_{22}^{-1}(T)|$ ,  $b_3 = |Q(I - \Gamma)|$ ,  $b_4 = \int_0^T |e^{As} B| ds$  and  $b_5 = \sup_{0 \leq t \leq T} |e^{At} B|$ . By exchanging the order of integration in  $q_1$  and  $q_2$ , it is easy to show

$$|q_1(t)|^2 \leq (b_1 b_2 b_3 b_4)^2 T \int_0^T \nu_i^2(s) ds, \quad |q_2(t)| \leq b_1 b_3 b_5 \int_0^t (t - \tau) |\nu_i(\tau)| d\tau,$$

which further gives

$$\int_0^T |q(t)|^2 dt \leq C_q \int_0^T |\nu_i(t)|^2 dt, \quad (29)$$

where  $C_q = 2(b_1 b_2 b_3 b_4)^2 T^2 + \frac{1}{6} (b_1 b_3 b_5)^2 T^4$ . For the case  $H = 0$ , (H2) holds whenever  $R > \gamma C_q I$ .

### 3.5 The solution of (P2)

Let  $\bar{u} \in C([0, T]; \mathbb{R}^{n_1})$  be fixed.

**Lemma 12** *Assume (H1)-(H2). Then (P2) has a unique optimal state-control pair of the form  $(x_i, m_i, p_i, \hat{u}_i)$  satisfying*

$$\begin{cases} dx_i = (Ax_i + B\hat{u}_i + Gm_i + \gamma p_i)dt + DdW_i, \\ \dot{m}_i = (A + G)m_i + B\bar{u} + \gamma p_i, \\ \dot{p}_i = -(A + G)^T p_i - (I - \Gamma)^T Q[\mathbb{E}x_i - (\Gamma m_i + \eta)], \end{cases} \quad (30)$$

where  $p_i(T) = H\mathbb{E}x_i(T)$ . Furthermore, the backward stochastic differential equation (BSDE)

$$\begin{cases} dy_i = \{-A^T y_i + Q[x_i - (\Gamma m_i + \eta)]\} dt + \zeta_i dW_i, \\ y_i(T) = -Hx_i(T) \end{cases} \quad (31)$$

has a unique solution  $(y_i, \zeta_i) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{2n})$  and

$$\hat{u}_i = R^{-1}B^T y_i. \quad (32)$$

*Proof.* Under (H2), by adapting Lemma 9 to the auxiliary control problem with cost functional  $\bar{J}_i(u_i) - \delta_0 \mathbb{E} \int_0^T |u_i|^2 dt$ , we can show that  $\bar{J}_i(u_i) - \delta_0 \mathbb{E} \int_0^T |u_i|^2 dt$  is convex in  $u_i$ . By the method in proving Lemma 6, we can further show that  $\bar{J}_i$  is strictly convex and coercive in  $u_i$ . Hence (P2) has a unique optimal state-control pair  $(x_i, m_i, p_i, \hat{u}_i)$  which minimizes  $\bar{J}_i(u_i)$ .

Given  $(x_i, m_i, p_i, \hat{u}_i)$ , (31) is a standard linear BSDE and so has a unique solution  $(y_i, \zeta_i)$ . Further define the BSDE

$$dy = \{-G^T y_i - (A + G)^T y - \Gamma^T Q[x_i - (\Gamma m_i + \eta)]\} dt + \zeta dW_i,$$

where  $y(T) = 0$ . It also has a unique solution  $(y, \zeta) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{2n})$ . It can be checked that

$$\frac{d}{dt}[\mathbb{E}(y + y_i) + p_i] = -(A + G)^T [\mathbb{E}(y + y_i) + p_i]$$

and  $\mathbb{E}(y(T) + y_i(T)) + p_i(T) = 0$ . So

$$\mathbb{E}(y_i + y) + p_i = 0 \quad (33)$$

for all  $t \in [0, T]$ .

Let  $\hat{u}_i$  be replaced by  $\hat{u}_i + \tilde{u}_i \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$  in (30), and the resulting solution be denoted by  $(x_i + \tilde{x}_i, m_i + \tilde{m}_i, p_i + \tilde{p}_i)$ , which exists and is unique by Theorem 7. It follows that

$$\begin{cases} \dot{\tilde{x}}_i = A\tilde{x}_i + B\tilde{u}_i + G\tilde{m}_i + \gamma\tilde{p}_i, \\ \dot{\tilde{m}}_i = (A + G)\tilde{m}_i + \gamma\tilde{p}_i, \\ \dot{\tilde{p}}_i = -(A + G)^T \tilde{p}_i - (I - \Gamma)^T Q(\mathbb{E}\tilde{x}_i - \Gamma\tilde{m}_i), \end{cases}$$

where  $\tilde{x}_i(0) = \tilde{m}_i(0) = 0$  and  $\tilde{p}_i(T) = H\mathbb{E}\tilde{x}_i(T)$ . The first variation of  $\bar{J}_i$  about  $\hat{u}_i$  satisfies

$$0 = \frac{\delta \bar{J}_i}{2} = \mathbb{E} \int_0^T \{(\tilde{x}_i - \Gamma\tilde{m}_i)^T Q[x_i - (\Gamma m_i + \eta)] + \tilde{u}_i^T R\hat{u}_i - \gamma\tilde{p}_i^T p_i\} dt + \mathbb{E}\tilde{x}_i^T(T)Hx_i(T). \quad (34)$$

By applying Ito's formula to  $\tilde{x}_i^T y_i$ , we obtain

$$\mathbb{E}\tilde{x}_i^T(T)y_i(T) - \mathbb{E}\tilde{x}_i^T(0)y_i(0) = \mathbb{E} \int_0^T \{\tilde{x}_i^T Q[x_i - (\Gamma m_i + \eta)] + y_i^T (B\tilde{u}_i + G\tilde{m}_i + \gamma\tilde{p}_i)\} dt.$$

Similarly,

$$\mathbb{E}\tilde{m}_i^T(T)y(T) - \mathbb{E}\tilde{m}_i^T(0)y(0) = \mathbb{E} \int_0^T \{ \gamma y^T \tilde{p}_i - \tilde{m}_i^T (G^T y_i + \Gamma^T Q[x_i - (\Gamma m_i + \eta)]) \} dt.$$

Therefore, adding up the two equations yields

$$- \mathbb{E}\tilde{x}_i^T(T)Hx_i(T) = \mathbb{E} \int_0^T \{ (\tilde{x}_i - \Gamma \tilde{m}_i)^T Q[x_i - (\Gamma m_i + \eta)] + y_i^T B \tilde{u}_i + \gamma(y + y_i)^T \tilde{p}_i \} dt. \quad (35)$$

By (34) and (35),

$$\mathbb{E} \int_0^T [\tilde{u}_i^T R \hat{u}_i - \gamma \tilde{p}_i^T p_i - \tilde{u}_i^T B^T y_i - \gamma \tilde{p}_i^T (y + y_i)] dt = 0.$$

Note that by (33),

$$\mathbb{E} \int_0^T \tilde{p}_i^T (p_i + y + y_i) dt = \int_0^T \tilde{p}_i^T [p_i + \mathbb{E}(y + y_i)] dt = 0.$$

Hence,

$$\mathbb{E} \int_0^T \tilde{u}_i^T (R \hat{u}_i - B^T y_i) dt = 0.$$

Since  $\tilde{u}_i \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$  is arbitrary, (32) follows.  $\square$

After substituting  $\hat{u}_i = R^{-1}B^T y_i$  into (30), we form the equation system

$$\begin{cases} dx_i = (Ax_i + BR^{-1}B^T y_i + Gm_i + \gamma p_i)dt + DdW_i, \\ \dot{m}_i = (A + G)m_i + B\bar{u} + \gamma p_i, \\ \dot{p}_i = -(A + G)^T p_i - (I - \Gamma)^T Q[\mathbb{E}x_i - (\Gamma m_i + \eta)], \\ dy_i = \{ -A^T y_i + Q[x_i - (\Gamma m_i + \eta)] \} dt + \zeta_i dW_i, \end{cases} \quad (36)$$

where  $x_i(0)$  is given,  $m_i(0) = m_0$ ,  $p_i(T) = H\mathbb{E}x_i(T)$ , and  $y_i(T) = -Hx_i(T)$ . This equation system consists of 2 forward equations and 2 backward equations. It is clear that the solution of the optimal control problem (P2) satisfies the above FBSDE. A natural question is whether this FBSDE's solution completely determines the optimal control. This is answered by the next theorem. Denote

$$S[0, T] = L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times C^1([0, T]; \mathbb{R}^{2n}) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{2n}).$$

**Theorem 13** Assume (H1)-(H2). Then the FBSDE (36) has a unique solution  $(x_i, m_i, p_i, y_i, \zeta_i) \in S[0, T]$  and the optimal control for (P2) is given by  $\hat{u}_i = R^{-1}B^T y_i$ .

*Proof.* We solve (P1) first and (P2) next to determine  $\hat{u}_i$ . By Lemma 12, we obtain  $(x_i, m_i, p_i, y_i, \zeta_i)$  to satisfy (30)-(31) and  $\hat{u}_i = R^{-1}B^T y_i$ . Obviously,  $(x_i, m_i, p_i, y_i, \zeta_i)$  satisfies (36).

We continue to show uniqueness. Suppose that  $(x_i, m_i, p_i, y_i, \zeta_i)$  and  $(x'_i, m'_i, p'_i, y'_i, \zeta'_i)$  are two solutions of (36). Define  $\tilde{u}_i = R^{-1}B^T y_i$  and  $u'_i = R^{-1}B^T y'_i$  which are both well-determined elements in  $L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$ . In particular, we have

$$\begin{cases} dx_i = (Ax_i + B\tilde{u}_i + Gm_i + \gamma p_i)dt + DdW_i, \\ \dot{m}_i = (A + G)m_i + B\bar{u} + \gamma p_i, \\ \dot{p}_i = -(A + G)^T p_i - (I - \Gamma)^T Q[\mathbb{E}x_i - (\Gamma m_i + \eta)], \\ dy_i = \{ -A^T y_i + Q[x_i - (\Gamma m_i + \eta)] \} dt + \zeta_i dW_i, \end{cases} \quad (37)$$

where  $x_i(0)$  is given,  $m_i(0) = m_0$ ,  $p_i(T) = H\mathbb{E}x_i(T)$ , and  $y_i(T) = -Hx_i(T)$ .

As in the proof of Lemma 12, we evaluate the first variation of  $\bar{J}_i(u_i)$  at  $(x_i, m_i, p_i, \tilde{u}_i)$  and can show  $\delta \bar{J}_i = 0$ . Since  $\bar{J}_i$  is convex, this zero first variation condition implies that  $\tilde{u}_i$  is an optimal control of (P2). By the same reasoning,  $u'_i$  is also an optimal control. By strict convexity, we have  $\tilde{u}_i = u'_i$ . Subsequently, we have  $(x_i, m_i, p_i) = (x'_i, m'_i, p'_i)$  by Theorem 7. This further implies  $(y_i, \zeta_i) = (y'_i, \zeta'_i)$ .  $\square$

## 4 The Solution of the Robust Game

Note that Theorem 13 determines the strategy of a representative agent when  $\bar{u}$  is fixed. Denote

$$x^{(N)} = \frac{1}{N} \sum_{i=1}^N x_i, \quad y^{(N)} = \frac{1}{N} \sum_{i=1}^N y_i, \quad m^{(N)} = \frac{1}{N} \sum_{i=1}^N m_i, \quad p^{(N)} = \frac{1}{N} \sum_{i=1}^N p_i. \quad (38)$$

By (36), we obtain

$$\begin{cases} dx^{(N)} = (Ax^{(N)} + BR^{-1}B^T y^{(N)} + Gm^{(N)} + \gamma p^{(N)}) dt + \frac{D}{N} \sum_{i=1}^N dW_i, \\ \frac{dm^{(N)}}{dt} = (A + G)m^{(N)} + B\bar{u} + \gamma p^{(N)}, \\ \frac{dp^{(N)}}{dt} = -(A + G)^T p^{(N)} - (I - \Gamma)^T Q [\mathbb{E}x^{(N)} - (\Gamma m^{(N)} + \eta)], \\ dy^{(N)} = \{-A^T y^{(N)} + Q[x^{(N)} - (\Gamma m^{(N)} + \eta)]\} dt + \frac{1}{N} \sum_{i=1}^N \zeta_i dW_i, \end{cases} \quad (39)$$

where  $x^{(N)}(0) = (1/N) \sum_{i=1}^N x_i(0)$ ,  $m^{(N)}(0) = m_0$ ,  $p^{(N)}(T) = H\mathbb{E}x^{(N)}(T)$ , and  $y^{(N)}(T) = -Hx^{(N)}(T)$ .

As an approximation to (39), we construct the following limiting system

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + BR^{-1}B^T \mathbf{y} + G\mathbf{m} + \gamma \mathbf{p}, \\ \dot{\mathbf{m}} = (A + G)\mathbf{m} + B\bar{u} + \gamma \mathbf{p}, \\ \dot{\mathbf{p}} = -(A + G)^T \mathbf{p} - (I - \Gamma)^T Q[\mathbf{x} - (\Gamma \mathbf{m} + \eta)], \\ \dot{\mathbf{y}} = -A^T \mathbf{y} + Q[\mathbf{x} - (\Gamma \mathbf{m} + \eta)], \end{cases} \quad (40)$$

where  $\mathbf{x}(0) = \mathbf{m}(0) = m_0$ ,  $\mathbf{p}(T) = H\mathbf{x}(T)$ , and  $\mathbf{y}(T) = -H\mathbf{x}(T)$ . This is a two point boundary value problem.

Note that  $\mathbf{y}$  is intended as an approximation of  $y^{(N)}$  when  $N \rightarrow \infty$ . The consistency requirement imposes

$$\bar{u} = R^{-1}B^T \mathbf{y}. \quad (41)$$

Under the condition (41), the first two equations in (40) coincide to give  $\mathbf{x} = \mathbf{m}$  for all  $t \in [0, T]$ . Consequently, we eliminate the equation of  $\mathbf{x}$  and introduce the new system

$$\begin{cases} \dot{\mathbf{m}} = (A + G)\mathbf{m} + BR^{-1}B^T \mathbf{y} + \gamma \mathbf{p}, \\ \dot{\mathbf{p}} = -(A + G)^T \mathbf{p} - (I - \Gamma)^T Q[\mathbf{m} - (\Gamma \mathbf{m} + \eta)], \\ \dot{\mathbf{y}} = -A^T \mathbf{y} + Q[\mathbf{m} - (\Gamma \mathbf{m} + \eta)], \end{cases} \quad (42)$$

where  $\mathbf{m}(0) = m_0$ ,  $\mathbf{p}(T) = H\mathbf{m}(T)$ , and  $\mathbf{y}(T) = -H\mathbf{m}(T)$ . This is still a two point boundary value problem. The next corollary follows from Theorem 13.

**Corollary 14** *Assume (H1)-(H2). Suppose that (42) has a unique solution  $(\mathbf{m}, \mathbf{p}, \mathbf{y}) \in C^1([0, T]; \mathbb{R}^{3n})$  and take  $\bar{u} = R^{-1}B^T \mathbf{y}$  in (36). Then (36) has a unique solution  $(x_i, m_i, p_i, y_i, \zeta_i) \in S[0, T]$ .  $\square$*

### 4.1 The special case of same initial conditions

Consider the special case where all agents have the same initial condition  $x_i(0) = m_0$  for all  $i \geq 1$ . The FBSDE (36) defines a mapping

$$\Lambda(\bar{u}) = R^{-1}B^T \mathbb{E}y_i,$$

where we take  $\bar{u} \in C([0, T]; \mathbb{R}^{n_1})$ . Clearly  $R^{-1}B^T \mathbb{E}y_i$  is a continuous  $\mathbb{R}^{n_1}$ -valued function of  $t \in [0, T]$ .

By the consistency requirement  $\bar{u} = \Lambda(\bar{u})$ , we set  $\bar{u} = R^{-1}B^T\mathbb{E}y_i$  in the second equation of (36) to obtain the equation system of the mean field game:

$$\begin{cases} dx_i = (Ax_i + BR^{-1}B^Ty_i + Gm_i + \gamma p_i)dt + DdW_i, \\ \dot{m}_i = (A + G)m_i + BR^{-1}B^T\mathbb{E}y_i + \gamma p_i, \\ \dot{p}_i = -(A + G)^Tp_i - (I - \Gamma)^TQ[\mathbb{E}x_i - (\Gamma m_i + \eta)], \\ dy_i = \{-A^Ty_i + Q[x_i - (\Gamma m_i + \eta)]\}dt + \zeta_i dW_i, \end{cases} \quad (43)$$

where  $x_i(0) = m_i(0) = m_0$ ,  $p_i(T) = H\mathbb{E}x_i(T)$ , and  $y_i(T) = -Hx_i(T)$ .

An interesting fact is that the existence and uniqueness of a solution to (43) is completely determined by the ODE system (42) without further using (H1)-(H2).

**Theorem 15** (43) has a unique solution  $(x_i, m_i, p_i, y_i, \zeta_i) \in S[0, T]$  if and only if (42) has a unique solution.

*Proof.* By Lemma B.1, (43) has a unique solution if and only if the FBSDE (B.1) has a unique solution. By Lemma B.2 and Lemma B.3-(iii), the FBSDE (B.1) has a unique solution if and only if (42) has a unique solution. The theorem follows.  $\square$

## 4.2 Existence of a solution to (42)

To study the existence and uniqueness of a solution to (42), we use a fixed point approach and introduce the equation system

$$\begin{cases} \dot{\mathbf{m}} = (A + G)\mathbf{m} + h + \gamma \mathbf{p}, \\ \dot{\mathbf{p}} = -(A + G)^T\mathbf{p} - \widehat{Q}\mathbf{m} + (I - \Gamma)^TQ\eta, \\ \dot{\mathbf{y}} = -A^T\mathbf{y} + Q[\mathbf{m} - (\Gamma\mathbf{m} + \eta)], \end{cases} \quad (44)$$

where  $h \in C([0, T]; \mathbb{R}^n)$ ,  $m(0) = m_0$ ,  $\mathbf{p}(T) = H\mathbf{m}(T)$ , and  $\mathbf{y}(T) = -H\mathbf{m}(T)$ . The next lemma identifies a sufficient condition for (44) to have a unique solution for any  $h \in C([0, T]; \mathbb{R}^n)$ .

**Lemma 16** Suppose that the Riccati equation

$$\dot{K} + K(A + G) + (A + G)^TK - \gamma K^2 - \widehat{Q} = 0, \quad K(T) = -H \quad (45)$$

has a unique solution on  $[0, T]$ . Then (44) defines a mapping from  $C([0, T]; \mathbb{R}^n)$  to itself:

$$\Lambda_1 : h \mapsto BR^{-1}B^T\mathbf{y}.$$

*Proof.* We write  $\mathbf{p} = -K\mathbf{m} + \phi$  for (44) and obtain the ODE

$$\dot{\phi} = -(A + G - \gamma K)\phi + Kh + (I - \Gamma)^TQ\eta, \quad \phi(T) = 0.$$

It follows that

$$\dot{\mathbf{m}} = (A + G - \gamma K)\mathbf{m} + h + \gamma \phi.$$

Let the fundamental solution matrices of the two ODEs

$$\dot{\phi} = (A + G - \gamma K)\phi, \quad \dot{\psi} = -(A + G - \gamma K)^T\psi$$

be  $\Phi(t, s)$  and  $\Psi(t, s)$ , respectively, with  $\Phi(s, s) = \Psi(s, s) = I$ . Then  $\Psi(t, s) = \Phi^T(s, t)$ . We obtain

$$\phi(t) = - \int_t^T \Psi(t, s_1)[K(s_1)h(s_1) + (I - \Gamma)^TQ\eta]ds_1.$$

This in turn gives

$$\begin{aligned}\mathbf{m}(t) &= \Phi(t, 0)m_0 + \int_0^t \Phi(t, s_1)h(s_1)ds_1 \\ &\quad - \gamma \int_0^t \Phi(t, s_2) \int_{s_2}^T \Psi(s_2, s_1)[K(s_1)h(s_1) + (I - \Gamma)^T Q\eta]ds_1 ds_2.\end{aligned}$$

We further solve

$$\mathbf{y}(t) = - \int_t^T e^{-A^T(t-s_3)} Q[(I - \Gamma)\mathbf{m}(s_3) - \eta]ds_3 - e^{-A^T(t-T)} H\mathbf{m}(T),$$

which implies  $\mathbf{y} \in C([0, T]; \mathbb{R}^n)$ . The lemma follows.  $\square$

To simplify the existence analysis for (42) in this section, we consider the case  $H = 0$ . Below  $\Upsilon_k$  denotes a continuous function of  $t$  which does not depend on  $h$  and can be easily determined. Consequently,

$$\begin{aligned}\mathbf{y}(t) &= - \int_t^T e^{-A^T(t-s_3)} Q[(I - \Gamma)\mathbf{m}(s_3) - \eta]ds_3 \\ &= - \int_t^T e^{-A^T(t-s_3)} Q(I - \Gamma)\mathbf{m}(s_3)ds_3 + \Upsilon_1(t) \\ &= - \int_t^T e^{-A^T(t-s_2)} Q(I - \Gamma) \int_0^{s_2} \Phi(s_2, s_1)h(s_1)ds_1 ds_2 \\ &\quad + \gamma \int_t^T e^{-A^T(t-s_3)} Q(I - \Gamma) \int_0^{s_3} \Phi(s_3, s_2) \int_{s_2}^T \Psi(s_2, s_1)K(s_1)h(s_1)ds_1 ds_2 ds_3 \\ &\quad + \Upsilon_2(t).\end{aligned}$$

Now we have

$$\begin{aligned}\Lambda_1(h)(t) &= BR^{-1}B^T \mathbf{y}(t) \\ &= -BR^{-1}B^T \int_t^T e^{-A^T(t-s_2)} Q(I - \Gamma) \int_0^{s_2} \Phi(s_2, s_1)h(s_1)ds_1 ds_2 \\ &\quad + \gamma BR^{-1}B^T \int_t^T e^{-A^T(t-s_3)} Q(I - \Gamma) \int_0^{s_3} \Phi(s_3, s_2) \int_{s_2}^T \Psi(s_2, s_1)K(s_1)h(s_1)ds_1 ds_2 ds_3 \\ &\quad + BR^{-1}B^T \Upsilon_2(t) \\ &=: \Lambda_0(h)(t) + BR^{-1}B^T \Upsilon_2(t).\end{aligned}$$

It is clear that  $\Lambda_0$  is from  $C([0, T]; \mathbb{R}^n)$  to itself.  $\square$

Define the constants

$$\begin{aligned}c_1 &= \max_{t \in [0, T]} |K(t)|, \quad c_2 = \max_{0 \leq t, s \leq T} |\Phi(t, s)|, \\ c_3 &= \max_{t \in [0, T]} \int_t^T |e^{A(s-t)}|s ds, \quad c_4 = \max_{t \in [0, T]} \int_t^T |e^{A(s-t)}|(Ts - \frac{s^2}{2})ds.\end{aligned}$$

Note that  $Ts - \frac{s^2}{2} \geq 0$  for  $s \in [0, T]$ . Denote  $|h| = \max_{t \in [0, T]} |h(t)|$ .

**Theorem 17** Assume  $H = 0$ . If

$$c_2 |BR^{-1}B^T| \cdot |Q(I - \Gamma)| (c_3 + \gamma c_1 c_2 c_4) < 1, \quad (46)$$

then (42) has a unique solution.



*Proof.* For each  $t$ ,

$$\begin{aligned}
|\Lambda_0(h)(t)| &\leq c_2|h| \cdot |BR^{-1}B^T| \cdot |Q(I - \Gamma)| \int_t^T |e^{A^T(s_2-t)}| s_2 ds_2 \\
&\quad + \gamma c_1 c_2^2 |h| \cdot |BR^{-1}B^T| \cdot |Q(I - \Gamma)| \int_t^T |e^{A^T(s_3-t)}| \int_0^{s_3} \int_{s_2}^T ds_1 ds_2 ds_3 \\
&= c_2|h| \cdot |BR^{-1}B^T| \cdot |Q(I - \Gamma)| \int_t^T |e^{A(s_2-t)}| s_2 ds_2 \\
&\quad + \gamma c_1 c_2^2 |h| \cdot |BR^{-1}B^T| \cdot |Q(I - \Gamma)| \int_t^T |e^{A(s_3-t)}| (Ts_3 - \frac{s_3^2}{2}) ds_3 \\
&= c_2|BR^{-1}B^T| \cdot |Q(I - \Gamma)| (c_3 + \gamma c_1 c_2 c_4).
\end{aligned}$$

Hence,  $\Lambda_1$  is a contraction and has a unique fixed point. So (42) has a unique solution.  $\square$

The constants  $c_1, \dots, c_4$  in (46) do not depend on  $BR^{-1}B^T$ . If  $BR^{-1}B^T$  is suitably small, (46) can be ensured.

**Example 18** Consider the system with parameters given by Example 4. Take  $T = 1.3$ . In analogue to (12), we can solve  $K(t)$  on  $[0, T]$  for (45). It can be shown that  $K(t) \leq 0$  for  $t \in [0, T]$  and  $|K(t)|$  attains its maximum on  $[0, T]$  at  $t = 0$ . We have  $K(0) = -0.171417$  which gives  $c_1 = 0.171417$ . So  $c_2 \leq e^{(A+G+|K(0)|)T} = 3.312961$ . Furthermore,

$$c_3 \leq \int_0^T e^{As} s ds = 1.318243, \quad c_4 \leq \int_0^T e^{As} (Ts - \frac{s^2}{2}) ds = 1.112937.$$

Subsequently,

$$c_2|BR^{-1}B^T| \cdot |Q(I - \Gamma)| (c_3 + \gamma c_1 c_2 c_4) \leq 0.861493.$$

So (46) holds.

**Remark 1** For the two-point boundary value problem, the contraction estimate in the fixed point method may be conservative and typically works on small time intervals for the solvability of (42) (see, e.g., Ch.1, Sec. 5, [40]).

We continue to derive another condition under which (42) is solvable without restriction to a small time horizon. To this end, we first rewrite (42) in the following form:

$$\begin{cases} \begin{pmatrix} \dot{m} \\ \dot{p} \\ \dot{y} \end{pmatrix} = \tilde{A} \begin{pmatrix} m \\ p \\ y \end{pmatrix} + \tilde{\eta}, \\ m(0) = m_0, \quad p(T) = Hm(T), \quad y(T) = -Hm(T), \end{cases} \quad (47)$$

where

$$\tilde{A} = \begin{pmatrix} A+G & \gamma & BR^{-1}B^T \\ -(I-\Gamma)^T Q(I-\Gamma) & -(A+G)^T & 0 \\ Q(1-\Gamma) & 0 & -A^T \end{pmatrix}, \quad \tilde{\eta} = \begin{pmatrix} 0 \\ (I-\Gamma)^T Q\eta \\ -Q\eta \end{pmatrix}.$$

Then, by the variation of constant formula, we have

$$\begin{pmatrix} m(t) \\ p(t) \\ y(t) \end{pmatrix} = \Theta(t) \begin{pmatrix} m_0 \\ \mu \\ \nu \end{pmatrix} + \Theta(t) \int_0^t \Theta^{-1}(s) \tilde{\eta} ds, \quad (48)$$

where  $\Theta(t) = e^{\tilde{A}t}$  and  $p, y$  have the initial conditions  $p(0) = \mu, y(0) = \nu$ . Noting the terminal condition in (47), now we present the following result.

**Proposition 19** [40, Ch.2, Sec. 3] *If for given  $T > 0$ ,  $\det(\tilde{\Theta}(T)) \neq 0$ , where*

$$\tilde{\Theta}(T) = \begin{pmatrix} -H & I & 0 \\ H & 0 & I \end{pmatrix} \Theta(T) \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix},$$

*then (42) has a unique solution on  $[0, T]$  for any initial value  $m_0$ .*

For illustration, we give the following example.

**Example 20** *Consider system (3)-(4) with all parameters being scalar-valued and  $\Gamma = 1$ ,  $H = 0$ . We calculate*

$$\mathcal{A} = \begin{pmatrix} A+G & -\gamma \\ 0 & -(A+G) \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} A+G & \gamma & R^{-1}B^2 \\ 0 & -(A+G) & 0 \\ 0 & 0 & -A \end{pmatrix},$$

*where  $\mathcal{A}$  is defined in Proposition 3. By direct computations, we obtain*

$$\det\{[(0, I)e^{\mathcal{A}t}(0, I)^T]\} = e^{-(A+G)t} > 0$$

*for all  $t \in [0, T]$ , which ensures (H1) by Proposition 3. Moreover,  $b_3 = 0$  gives  $C_q = 0$  in (29) so that (H2) always holds true for  $R > 0$ . Finally,  $\det(\tilde{\Theta}(t)) = e^{-(2A+G)t} > 0$ , and subsequently, (42) has a unique solution on any interval  $[0, T]$ . To summarize, (H1), (H2) and the solvability of (42) are all satisfied by the system.*

Note that the solvability of (42) in Example 20 does not depend on the value of  $R^{-1}B^2$ , which is different from the condition in Theorem 17.

## 5 Error Estimate of the Mean Field Approximation

We suppose that (42) has a unique solution  $(\mathbf{m}, \mathbf{p}, \mathbf{y})$  and accordingly take  $\bar{u}$  in (36) as

$$\bar{u}^* = R^{-1}B^T \mathbf{y}. \quad (49)$$

The FBSDE system (36) now becomes

$$\begin{cases} dx_i = (Ax_i + BR^{-1}B^T y_i + Gm_i + \gamma p_i)dt + DdW_i, \\ \dot{m}_i = (A+G)m_i + B\bar{u}^* + \gamma p_i, \\ \dot{p}_i = -(A+G)^T p_i - (I-\Gamma)^T Q[\mathbb{E}x_i - (\Gamma m_i + \eta)], \\ dy_i = \{-A^T y_i + Q[x_i - (\Gamma m_i + \eta)]\} dt + \zeta_i dW_i, \end{cases} \quad (50)$$

where  $x_i(0)$  is given,  $m_i(0) = m_0$ ,  $p_i(T) = H\mathbb{E}x_i(T)$ , and  $y_i(T) = -Hx_i(T)$ . By Corollary 14, this FBSDE has a unique solution. In the game of  $N$  players, let  $y_i$  be solved from (50) and denote the control for  $\mathcal{A}_i$  by

$$\hat{u}_i = R^{-1}B y_i, \quad 1 \leq i \leq N, \quad (51)$$

which is a well defined process in  $L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$ .

For  $\hat{u}^{(N)} = (1/N) \sum_{i=1}^N \hat{u}_i$ , we aim to estimate

$$\mathbb{E}|\hat{u}^{(N)}(t) - \bar{u}^*(t)|^2.$$

Note that  $\hat{u}_1, \dots, \hat{u}_N$  are independent, but they are not necessarily with the same distribution due to possibly different initial states of the agents. This fact will somehow complicate our error estimate. The key result of this section is the following theorem.

**Theorem 21** Assume that (H1)-(H2) hold and that (42) has a unique solution. We have

$$\sup_{0 \leq t \leq T} \mathbb{E} |\hat{u}^{(N)} - \bar{u}^*|^2 = O(1/N) + O(|x^{(N)}(0) - m_0|^2),$$

where  $x^{(N)}(0) = (1/N) \sum_{i=1}^N x_i(0)$ .  $\square$

The proof of Theorem 21 is provided in the remaining part of this section. To do this, we need to prove some lemmas under the assumption of the theorem. Recalling (38), we take  $\bar{u} = \bar{u}^*$  in (39) to write

$$\begin{cases} dx^{(N)} = (Ax^{(N)} + BR^{-1}B^T y^{(N)} + Gm^{(N)} + \gamma p^{(N)}) dt + \frac{D}{N} \sum_{i=1}^N dW_i, \\ \frac{dm^{(N)}}{dt} = (A + G)m^{(N)} + B\bar{u}^* + \gamma p^{(N)}, \\ \frac{dp^{(N)}}{dt} = -(A + G)^T p^{(N)} - (I - \Gamma)^T Q [\mathbb{E}x^{(N)} - (\Gamma m^{(N)} + \eta)], \\ dy^{(N)} = \{-A^T y^{(N)} + Q[x^{(N)} - (\Gamma m^{(N)} + \eta)]\} dt + \frac{1}{N} \sum_{i=1}^N \zeta_i dW_i, \end{cases} \quad (52)$$

where  $x^{(N)}(0) = (1/N) \sum_{i=1}^N x_i(0)$ ,  $m^{(N)}(0) = m_0$ ,  $p^{(N)}(T) = H\mathbb{E}x^{(N)}(T)$ , and  $y^{(N)}(T) = -Hx^{(N)}(T)$ .

Denote the ODE system

$$\begin{cases} \dot{\mathbf{x}}_N = A\mathbf{x}_N + BR^{-1}B^T \mathbf{y}_N + G\mathbf{m}_N + \gamma \mathbf{p}_N, \\ \dot{\mathbf{m}}_N = (A + G)\mathbf{m}_N + B\bar{u}^* + \gamma \mathbf{p}_N, \\ \dot{\mathbf{p}}_N = -(A + G)^T \mathbf{p}_N - (I - \Gamma)^T Q[\mathbf{x}_N - (\Gamma \mathbf{m}_N + \eta)], \\ \dot{\mathbf{y}}_N = -A^T \mathbf{y}_N + Q[\mathbf{x}_N - (\Gamma \mathbf{m}_N + \eta)], \end{cases} \quad (53)$$

where  $\mathbf{x}_N(0) = (1/N) \sum_{i=1}^N x_i(0)$ ,  $\mathbf{m}_N(0) = m_0$ ,  $\mathbf{p}_N(T) = H\mathbf{x}_N(T)$ , and  $\mathbf{y}_N(T) = -H\mathbf{x}_N(T)$ . The initial condition  $\mathbf{x}_N(0)$  is different from that of (40).

**Lemma 22** (53) has a unique solution which can be denoted as

$$(\mathbf{x}_N, \mathbf{m}_N, \mathbf{p}_N, \mathbf{y}_N) = (\mathbb{E}x^{(N)}, m^{(N)}, p^{(N)}, \mathbb{E}y^{(N)}).$$

*Proof.* Existence follows by taking expectation in (52). To show uniqueness, suppose that (53) has two different solutions  $(\mathbf{x}_N, \mathbf{m}_N, \mathbf{p}_N, \mathbf{y}_N)$  and  $(\mathbf{x}'_N, \mathbf{m}'_N, \mathbf{p}'_N, \mathbf{y}'_N)$ . Then for any  $\lambda \in \mathbb{R}$ ,

$$(x_i, m_i, p_i, y_i, \zeta_i) + \lambda(\mathbf{x}_N - \mathbf{x}'_N, \mathbf{m}_N - \mathbf{m}'_N, \mathbf{p}_N - \mathbf{p}'_N, \mathbf{y}_N - \mathbf{y}'_N, 0)$$

satisfies (36), which is a contradiction to Theorem 13. Uniqueness follows.  $\square$

**Lemma 23** We have

$$\sup_{0 \leq t \leq T} \left( \mathbb{E}|x^{(N)} - \mathbb{E}x^{(N)}|^2 + \mathbb{E}|y^{(N)} - \mathbb{E}y^{(N)}|^2 \right) = O(1/N).$$

*Proof.* Define

$$(\theta_1, \theta_2) = (x^{(N)} - \mathbb{E}x^{(N)}, y^{(N)} - \mathbb{E}y^{(N)}).$$

By (52), (53) and Lemma 22,

$$\begin{cases} d\theta_1 = (A\theta_1 + BR^{-1}B^T \theta_2)dt + \frac{D}{N} \sum_{i=1}^N dW_i, \\ d\theta_2 = (-A^T \theta_2 + Q\theta_1)dt + \frac{1}{N} \sum_{i=1}^N \zeta_i dW_i, \end{cases}$$

where  $\theta_1(0) = 0$  and  $\theta_2(T) = -H\theta_1(T)$ .

Let  $P$  be the solution of the Riccati equation

$$\dot{P} + A^T P + P A - P B R^{-1} B^T P + Q = 0, \quad P(T) = H.$$

Denote  $\theta_2 = -P\theta_1 + \psi$ , where  $\psi(T) = 0$ . This gives the equation

$$d\psi = -(A - B R^{-1} B^T P)^T \psi dt + \frac{1}{N} \sum_{i=1}^N (P D + \zeta_i) dW_i,$$

where  $\psi(T) = 0$ . There is a unique solution  $\psi = 0$  for  $t \in [0, T]$ . This implies

$$d\theta_1 = (A - B R^{-1} B^T P)\theta_1 dt + \frac{D}{N} \sum_{i=1}^N dW_i.$$

Hence,  $\sup_{0 \leq t \leq T} \mathbb{E}|\theta_1(t)|^2 = O(1/N)$ . The lemma follows since  $\theta_2 = -P\theta_1$ .  $\square$

When  $(\mathbf{m}, \mathbf{p}, \mathbf{y})$  is a unique solution of (42), it can be shown that  $(\mathbf{x}, \mathbf{m}, \mathbf{y}, \mathbf{p}) := (\mathbf{m}, \mathbf{m}, \mathbf{y}, \mathbf{p})$  is a unique solution of (40) under the condition (41).

**Lemma 24** *We have*

$$\sup_{0 \leq t \leq T} [|\mathbf{x}_N - \mathbf{x}| + |\mathbf{m}_N - \mathbf{m}| + |\mathbf{p}_N - \mathbf{p}| + |\mathbf{y}_N - \mathbf{y}|] = O(|x^{(N)}(0) - m_0|).$$

*Proof.* Consider

$$\begin{cases} \dot{h}_1 = A h_1 + B R^{-1} B^T h_4 + G h_2 + \gamma h_3, \\ \dot{h}_2 = (A + G) h_2 + \gamma h_3, \\ \dot{h}_3 = -(A + G)^T h_3 - (I - \Gamma)^T Q (h_1 - \Gamma h_2), \\ \dot{h}_4 = -A^T h_4 + Q (h_1 - \Gamma h_2), \end{cases} \quad (54)$$

where  $h_1(0)$  is given,  $h_2(0) = 0$ ,  $h_3(T) = H h_1(T)$ , and  $h_4(T) = -H h_1(T)$ . It is constructed as a homogeneous version of (53). We claim that (54) has a unique solution for any given value of  $h_1(0)$ . If this were not true, there would exist  $h(0)$  such that (54) has multiple solutions which, in turn, can be used to construct multiple solutions to (53). This would give a contradiction to Lemma 22.

It is clear that

$$(\mathbf{x}_N - \mathbf{x}, \mathbf{m}_N - \mathbf{m}, \mathbf{p}_N - \mathbf{p}, \mathbf{y}_N - \mathbf{y}) =: (h_1, h_2, h_3, h_4),$$

is a solution of (54) with  $h_1(0) = \mathbf{x}_N(0) - m_0$ .

Let  $e_1, \dots, e_n$  be a canonical basis of  $\mathbb{R}^n$ . For  $h_1(0) = e_k$ , we obtain a solution of (54), denoted by  $h^k = (h_1^k, h_2^k, h_3^k, h_4^k)$ . Let  $(z)_k$  be the  $k$ th component of a vector  $z$ . We may uniquely denote  $(\mathbf{x}_N - \mathbf{x}, \mathbf{m}_N - \mathbf{m}, \mathbf{p}_N - \mathbf{p}, \mathbf{y}_N - \mathbf{y})$  as a linear combination of  $h^1, \dots, h^n$ :

$$(\mathbf{x}_N - \mathbf{x}, \mathbf{m}_N - \mathbf{m}, \mathbf{p}_N - \mathbf{p}, \mathbf{y}_N - \mathbf{y}) = \sum_{k=1}^n (x_N(0) - m_0)_k (h_1^k, h_2^k, h_3^k, h_4^k).$$

The lemma follows readily.  $\square$

**Proof of Theorem 21.** For  $\bar{u} = \bar{u}^*$ , we write  $\hat{u}^{(N)} = R^{-1} B^T (1/N) \sum_{i=1}^N y_i = R^{-1} B^T y^{(N)}$ . We have

$$\begin{aligned} |\hat{u}^{(N)} - \bar{u}^*|^2 &= \mathbb{E} |R^{-1} B^T (y^{(N)} - \mathbf{y})|^2 \\ &\leq C \mathbb{E} |y^{(N)} - \mathbf{y}|^2 \\ &= C E |y^{(N)} - \mathbb{E} y^{(N)} + \mathbb{E} y^{(N)} - \mathbf{y}|^2 \\ &\leq C(1/N) + C |\mathbf{y}_N - \mathbf{y}|^2 \\ &= O(1/N) + O(|x^{(N)}(0) - m_0|^2). \end{aligned}$$

The second inequality follows from Lemmas 22 and 23, and the last step follows from Lemma 24.  $\square$

## 6 Robust Nash Equilibrium

Throughout this section, we assume that (42) has a unique solution and take  $\bar{u} = \bar{u}^*$  determined by (49). For  $f \in L^2(0, T; \mathbb{R}^n)$  and  $u_i \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$ ,  $1 \leq i \leq N$ , recall the worst case cost

$$J_i^{\text{wo}}(u_i, u_{-i}) = \sup_{f \in L^2(0, T; \mathbb{R}^n)} J_i(u_i, u_{-i}, f).$$

It is clear that for each  $i$  and any  $(u_i, u_{-i})$ ,  $\sup_f J_i(u_i, u_{-i}, f) \geq 0$ .

Consider the set of strategies  $(\hat{u}_i, \hat{u}_{-i})$  given by (51) for a population of  $N$  players with dynamics (3). It should be emphasized that we only use (50)-(51) to make a well defined process  $\hat{u}_i$  in  $L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$  which should not be understood as a feedback strategy. The main result of this section is the next theorem which characterizes the performance of this set of strategies.

**Theorem 25** *Assume (i) (H1)-(H2) hold; (ii)  $\sup_{i \geq 0} |x_i(0)| \leq M_0$  where  $M_0$  does not depend on  $N$ ; (iii) (42) has a unique solution. Then the set of strategies  $(\hat{u}_1, \dots, \hat{u}_N)$  given by (51) is a robust  $\varepsilon_N$ -Nash equilibrium for the  $N$  players, i.e.,*

$$J_i^{\text{wo}}(\hat{u}_i, \hat{u}_{-i}) - \varepsilon_N \leq \inf_{u_i \in \mathcal{U}} J_i^{\text{wo}}(u_i, \hat{u}_{-i}) \leq J_i^{\text{wo}}(\hat{u}_i, \hat{u}_{-i}), \quad (55)$$

where  $0 \leq \varepsilon_N = O(1/\sqrt{N} + |x^{(N)}(0) - m_0|)$  and  $x^{(N)}(0) = (1/N) \sum_{j=1}^N x_j(0)$ .  $\square$

The rest part of this section is devoted to the proof of Theorem 25. For any given  $f \in L^2(0, T; \mathbb{R}^n)$ , denote the state processes of (3) corresponding to  $(\hat{u}_i, \hat{u}_{-i}, f)$  by  $\hat{x}_j$ ,  $1 \leq j \leq N$ , and  $\hat{x}^{(N)} = (1/N) \sum_{j=1}^N \hat{x}_j$ . Denote

$$\dot{\bar{m}} = (A + G)\bar{m} + B\bar{u}^* + f, \quad \bar{m}(0) = m_0 \quad (56)$$

All subsequent lemmas are proved under the assumptions of Theorem 25.

**Lemma 26** *We have*

$$\sup_{0 \leq t \leq T, f} \mathbb{E}|\hat{x}^{(N)} - \bar{m}|^2 \leq C(1/N + |x^{(N)}(0) - m_0|^2).$$

*Proof.* Note that

$$d\hat{x}^{(N)} = [(A + G)\hat{x}^{(N)} + B\hat{u}^{(N)} + f]dt + (D/N) \sum_{i=1}^N dW_i.$$

Therefore,

$$d(\hat{x}^{(N)} - \bar{m}) = [(A + G)(\hat{x}^{(N)} - \bar{m}) + B(\hat{u}^{(N)} - \bar{u}^*)]dt + (D/N) \sum_{i=1}^N dW_i.$$

By linear SDE estimates,

$$\begin{aligned} \mathbb{E}|\hat{x}^{(N)}(t) - \bar{m}(t)|^2 &\leq C|x^{(N)}(0) - m_0|^2 + C/N \\ &\quad + C\mathbb{E} \int_0^t |\hat{u}^{(N)}(\tau) - \bar{u}^*(\tau)|^2 d\tau. \end{aligned}$$

By Theorem 21, the lemma follows.  $\square$

**Lemma 27** *There exists a constant  $\hat{C}_0$  independent of  $N$  such that*

$$\max_{1 \leq i \leq N} \sup_f J_i(\hat{u}_i, \hat{u}_{-i}, f) \leq \hat{C}_0.$$

*Proof.* Denote

$$dx'_i = (Ax'_i + B\hat{u}_i + G\bar{m} + f)dt + DdW_i, \quad (57)$$

where  $x'_i(0) = x_i(0)$ . By Lemma 26, it is easy to show

$$\sup_{0 \leq t \leq T, f} \mathbb{E}|\hat{x}_i(t) - x'_i(t)|^2 \leq C(1/N + |x^{(N)}(0) - m_0|^2).$$

We have

$$\begin{aligned} J_i(\hat{u}_i, \hat{u}_{-i}, f) &\leq \bar{J}_i(\hat{u}_i, f) + \mathbb{E} \int_0^T |(\hat{x}_i - x'_i) + \Gamma(\bar{m} - \hat{x}^{(N)})|_Q^2 dt + \mathbb{E}|\hat{x}_i(T) - x'_i(T)|_H^2 \\ &\quad + 2\mathbb{E} \int_0^T [x'_i - (\Gamma\bar{m} + \eta)]^T Q[(\hat{x}_i - x'_i) + \Gamma(\bar{m} - \hat{x}^{(N)})] dt \\ &\quad + 2\mathbb{E}[x_i'^T(T)H(\hat{x}_i(T) - x'_i(T))]. \end{aligned} \quad (58)$$

Combining Lemma 6 with condition (ii) in Theorem 25, we obtain

$$\bar{J}_i(\hat{u}_i, f) \leq C - (\epsilon_0/2)\|f\|_{L^2}^2 \quad (59)$$

for  $\epsilon_0 > 0$ , where  $C$  does not depend on  $(i, N)$ . Since neither  $\hat{x}_i - x'_i$  nor  $\bar{m} - \hat{x}^{(N)}$  depend on  $f$ , there exists a constant  $C_1$  such that

$$\begin{aligned} &\left| \mathbb{E} \int_0^T [x'_i - (\Gamma\bar{m} + \eta)]^T Q[(\hat{x}_i - x'_i) + \Gamma(\bar{m} - \hat{x}^{(N)})] dt \right| \\ &\leq C_1 \left( \mathbb{E} \int_0^T |x'_i - (\Gamma\bar{m} + \eta)|_Q^2 dt \right)^{1/2} \\ &\leq C_2(1 + \|f\|_{L^2}^2)^{1/2} \\ &\leq C_3 + (\epsilon_0/16)\|f\|_{L^2}^2, \end{aligned} \quad (60)$$

where the second inequality follows from elementary estimates based on the solutions of (56) and (57). Similarly,

$$\mathbb{E}[x_i'^T(T)H(\hat{x}_i(T) - x'_i(T))] \leq C_4 + (\epsilon_0/16)\|f\|_{L^2}^2. \quad (61)$$

Finally combining (58)-(61) with Lemma 26 leads to

$$J_i(\hat{u}_i, \hat{u}_{-i}, f) \leq C - (\epsilon_0/4)\|f\|_{L^2}^2.$$

The lemma follows. □

Consider the set of strategies  $(u_i, \hat{u}_{-i})$  and the corresponding state processes

$$dx_i = (Ax_i + Bu_i + Gx^{(N)} + f)dt + DdW_i, \quad (62)$$

$$dx_j = (Ax_j + B\hat{u}_j + Gx^{(N)} + f)dt + DdW_j, \quad 1 \leq j \leq N, \ j \neq i. \quad (63)$$

**Lemma 28** *If  $u_i$  in (62) satisfies  $\sup_f J_i(u_i, \hat{u}_{-i}, f) \leq \hat{C}_0$ , there exists  $\hat{C}_1$  independent of  $N$  such that*

$$\mathbb{E} \int_0^T |u_i(t)|^2 dt \leq \hat{C}_1. \quad (64)$$

*Proof.* Suppose  $\sup_f J_i(u_i, \hat{u}_{-i}, f) \leq \hat{C}_0$ . Then for any  $f$ ,

$$\mathbb{E} \int_0^T \left( |x_i - (\Gamma x^{(N)} + \eta)|_Q^2 + u_i^T R u_i - \frac{1}{\gamma} |f(t)|^2 \right) dt + \mathbb{E}[x_i^T(T) H x_i(T)] \leq \hat{C}_0,$$

where  $(x_1, \dots, x_N)$  is generated by  $(u_i, \hat{u}_{-i})$  and  $f$ . Taking  $f = 0$ , we obtain

$$\mathbb{E} \int_0^T \left( |x_i - (\Gamma x^{(N)} + \eta)|_Q^2 + u_i^T R u_i \right) dt \leq \hat{C}_0.$$

Therefore, (64) holds.  $\square$

Let  $\mathcal{U}_{\hat{C}_1}$  denote the set of processes  $u_i \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$  which satisfy (64). For (62)-(63), denote  $x^{(N)} = (1/N) \sum_{j=1}^N x_j$ .

**Lemma 29** *Suppose  $u_i \in \mathcal{U}_{\hat{C}_1}$  in (62). Then*

$$\sup_{0 \leq t \leq T, f, u_i \in \mathcal{U}_{\hat{C}_1}} \mathbb{E} |x^{(N)}(t) - \bar{m}(t)|^2 = O(1/N + |x^{(N)}(0) - m_0|^2).$$

*Proof.* Rewrite (62) in the form

$$dx_i = [Ax_i + B\hat{u}_i + Gx^{(N)} + f]dt + B(u_i - \hat{u}_i)dt + DdW_i. \quad (65)$$

By (63) and (65),

$$dx^{(N)} = [(A + G)x^{(N)} + B\hat{u}^{(N)} + f]dt + \frac{B}{N}(u_i - \hat{u}_i)dt + \frac{D}{N} \sum_{j=1}^N dW_j,$$

which combined with (56) gives

$$\begin{aligned} d(x^{(N)} - \bar{m}) &= [(A + G)(x^{(N)} - \bar{m}) + B(\hat{u}^{(N)} - \bar{u}^*)]dt \\ &\quad + \frac{B}{N}(u_i - \hat{u}_i)dt + \frac{D}{N} \sum_{j=1}^N dW_j. \end{aligned}$$

By Theorem 21 and the fact  $\mathbb{E} \int_0^T |u_i - \hat{u}_i|^2 \leq C$  for all  $u_i \in \mathcal{U}_{\hat{C}_1}$ , where the constants  $C$  do not depend on  $(f, u_i)$ , elementary SDE estimates lead to

$$\sup_{0 \leq t \leq T, f} \mathbb{E} |x^{(N)}(t) - \bar{m}(t)|^2 \leq C(1/N + |x^{(N)}(0) - m_0|^2),$$

where  $C$  does not depend on  $u_i$ . The lemma follows.  $\square$

**Lemma 30** *For each  $u_i \in \mathcal{U}_{\hat{C}_1}$ ,  $\sup_f J_i(u_i, \hat{u}_{-i}, f)$  is finite and attained by some  $f$  depending on  $u_i$  and so denoted as  $f_{u_i}$ . Moreover,*

$$\sup_{u_i \in \mathcal{U}_{\hat{C}_1}} \left| \sup_f J_i(u_i, \hat{u}_{-i}, f) - \bar{J}_i(u_i, \hat{f}_{u_i}) \right| = O(1/\sqrt{N} + |x^{(N)}(0) - m(0)|),$$

where  $\hat{f}_{u_i}$  is determined by Theorem 7 for the given  $u_i$ .

*Proof.* Note that we have

$$\begin{aligned} dx_i &= [Ax_i + Bu_i + G\bar{m} + G(x^{(N)} - \bar{m}) + f]dt + DdW_i, \\ \dot{\bar{m}} &= (A + G)\bar{m} + B\bar{u}^* + f, \end{aligned} \quad (66)$$

where  $\bar{m}(0) = m_0$ . Define the auxiliary process

$$dx_i^\dagger = (Ax_i^\dagger + Bu_i + G\bar{m} + f)dt + DdW_i,$$

where  $x_i^\dagger(0) = x_i(0)$  and  $(u_i, f, W_i)$  is the same as in (66). By Lemma 29, it is easy to show

$$\sup_{0 \leq t \leq T, f} \mathbb{E}|x_i(t) - x_i^\dagger(t)|^2 = O(1/N + |x^{(N)}(0) - m(0)|^2). \quad (67)$$

We have the relation

$$\begin{aligned} |x_i - (\Gamma x^{(N)} + \eta)|_Q^2 &= |x_i^\dagger - (\Gamma \bar{m} + \eta)|_Q^2 + |(x_i - x_i^\dagger) + \Gamma(\bar{m} - x^{(N)})|_Q^2 \\ &\quad + 2[x_i^\dagger - (\Gamma \bar{m} + \eta)]^T Q[(x_i - x_i^\dagger) + \Gamma(\bar{m} - x^{(N)})]. \end{aligned}$$

The cost can be rewritten as

$$\begin{aligned} J_i(u_i, \hat{u}_{-i}, f) &= \bar{J}_i(u_i, f) + \mathbb{E} \int_0^T |(x_i - x_i^\dagger) + \Gamma(\bar{m} - x^{(N)})|_Q^2 dt \\ &\quad + \mathbb{E} \left[ |x_i(T) - x_i^\dagger(T)|_H^2 \right] \\ &\quad + 2\mathbb{E} \int_0^T \left[ x_i^\dagger - (\Gamma \bar{m} + \eta) \right]^T Q \left[ (x_i - x_i^\dagger) + \Gamma(\bar{m} - x^{(N)}) \right] dt \\ &\quad + 2\mathbb{E} \left[ (x_i^\dagger(T))^T H(x_i(T) - x_i^\dagger(T)) \right] \end{aligned} \quad (68)$$

$$\begin{aligned} &\leq \bar{J}_i(u_i, f) + C \left( 1/N + |x^{(N)}(0) - m_0|^2 \right) \\ &\quad + 2\mathbb{E} \int_0^T \left[ x_i^\dagger - (\Gamma \bar{m} + \eta) \right]^T Q \left[ (x_i - x_i^\dagger) + \Gamma(\bar{m} - x^{(N)}) \right] dt \\ &\quad + 2\mathbb{E} \left[ (x_i^\dagger(T))^T H(x_i(T) - x_i^\dagger(T)) \right], \end{aligned} \quad (69)$$

where the inequality follows from Lemma 29 and (67). Note that neither  $x_i - x_i^\dagger$  nor  $\bar{m} - x^{(N)}$  in (68) depend on  $f$ . The terms  $x_i^\dagger$  and  $x_i^\dagger - (\Gamma \bar{m} + \eta)$  are affine in  $f$ , and  $-\bar{J}_i(u_i, f)$  is convex in  $f$  by Lemma 6. Consequently, it follows from (68) that  $-J_i(u_i, \hat{u}_{-i}, f)$  is convex in  $f$ . For  $u_i \in \mathcal{U}_{\hat{C}_1}$ , in analogue to (59), we obtain

$$\bar{J}_i(u_i, f) \leq C - (\epsilon_0/2)\|f\|_{L^2}^2, \quad (70)$$

where  $C$  does not depend on  $u_i$ . We have

$$\begin{aligned} &\left| \mathbb{E} \int_0^T \left[ x_i^\dagger - (\Gamma \bar{m} + \eta) \right]^T Q \left[ (x_i - x_i^\dagger) + \Gamma(\bar{m} - x^{(N)}) \right] dt \right| \\ &\leq \left\{ \mathbb{E} \int_0^T |x_i^\dagger - (\Gamma \bar{m} + \eta)|_Q^2 dt \right\}^{1/2} \cdot \left\{ \mathbb{E} \int_0^T |(x_i - x_i^\dagger) + \Gamma(\bar{m} - x^{(N)})|_Q^2 dt \right\}^{1/2} \\ &\leq C \left( 1/\sqrt{N} + |x^{(N)}(0) - m_0| \right) (1 + \|f\|_{L^2}^2)^{1/2} \\ &\leq C + (\epsilon_0/16)\|f\|_{L^2}^2. \end{aligned}$$



Similarly,

$$\left| \mathbb{E} \left[ (x_i^\dagger(T))^T H(x_i(T) - x_i^\dagger(T)) \right] \right| \leq C + (\epsilon_0/16) \|f\|_{L^2}^2.$$

Hence, (69) gives

$$J_i(u_i, \hat{u}_{-i}, f) \leq C - (\epsilon_0/4) \|f\|_{L^2}^2, \quad (71)$$

where  $C$  does not depend on  $(N, u_i)$ . So for given  $u_i \in \mathcal{U}_{\hat{C}_1}$ ,  $J_i(u_i, \hat{u}_{-i}, f)$  attains a finite supreme at some  $f_{u_i}$  since it is a continuous functional of  $f$ , and by (71) we may further find a constant  $\hat{C}_2$  such that

$$\sup_{u_i \in \mathcal{U}_{\hat{C}_1}} \|f_{u_i}\|_{L^2} \leq \hat{C}_2. \quad (72)$$

By (69),

$$\begin{aligned} J_i(u_i, \hat{u}_{-i}, f) &\leq \bar{J}_i(u_i, f) + C(1/N + |x^{(N)}(0) - m_0|^2) \\ &\quad + C \left( 1/N + |x^{(N)}(0) - m_0|^2 \right)^{1/2} \left( \mathbb{E} \int_0^T |x_i^\dagger - (\Gamma \bar{m} + \eta)|_Q^2 dt \right)^{1/2} \\ &\quad + C \left( 1/N + |x^{(N)}(0) - m_0|^2 \right)^{1/2} \left( \mathbb{E} |x_i^\dagger(T)|^2 \right)^{1/2}. \end{aligned} \quad (73)$$

Now for  $u_i \in \mathcal{U}_{\hat{C}_1}$  and the resulting  $f_{u_i}$  satisfying (72), we further obtain

$$\mathbb{E} |x_i^\dagger(T)|^2 + \mathbb{E} \int_0^T |x_i^\dagger - (\Gamma \bar{m} + \eta)|_Q^2 dt \leq C.$$

For  $u_i \in \mathcal{U}_{\hat{C}_1}$ , (73) gives

$$\begin{aligned} \sup_f J_i(u_i, \hat{u}_{-i}, f) &\leq \bar{J}_i(u_i, f_{u_i}) + C(1/\sqrt{N} + |x^{(N)}(0) - m_0|) \\ &\leq \bar{J}_i(u_i, \hat{f}_{u_i}) + C(1/\sqrt{N} + |x^{(N)}(0) - m_0|), \end{aligned}$$

where  $\hat{f}_{u_i}$  is determined by Theorem 7. Due to (70),

$$\sup_{u_i \in \mathcal{U}_{\hat{C}_1}} \|\hat{f}_{u_i}\|_{L^2} \leq C \quad (74)$$

for some constant  $C$ . By (74) and the method in (68), we similarly derive

$$J_i(u_i, \hat{u}_{-i}, \hat{f}_{u_i}) \geq \bar{J}_i(u_i, \hat{f}_{u_i}) - C(1/\sqrt{N} + |x^{(N)}(0) - m_0|).$$

Hence, for all  $u_i \in \mathcal{U}_{\hat{C}_1}$ ,

$$\sup_f J_i(u_i, \hat{u}_{-i}, f) \geq \bar{J}_i(u_i, \hat{f}_{u_i}) - C(1/\sqrt{N} + |x^{(N)}(0) - m_0|).$$

The constant  $C$  in various places does not depend on  $u_i$ . The lemma follows.  $\square$

**Proof of Theorem 25.** It suffices to show the first inequality by checking  $u_i \in \mathcal{U}_{\hat{C}_1}$ . By Lemma 30, we have

$$\begin{aligned} \sup_f J_i(u_i, \hat{u}_{-i}, f) &\geq \bar{J}_i(u_i, \hat{f}_{u_i}) - C_1(1/\sqrt{N} + |x^{(N)}(0) - m_0|) \\ &\geq \bar{J}_i(\hat{u}_i, \hat{f}_{\hat{u}_i}) - C_1(1/\sqrt{N} + |x^{(N)}(0) - m_0|). \end{aligned} \quad (75)$$

On the other hand, by taking the particular control  $\hat{u}_i$  in Lemma 30,

$$\sup_f J_i(\hat{u}_i, \hat{u}_{-i}, f) \leq \bar{J}_i(\hat{u}_i, \hat{f}_{\hat{u}_i}) + C_2(1/\sqrt{N} + |x^{(N)}(0) - m_0|). \quad (76)$$

Subsequently, (75) and (76) imply

$$\sup_f J_i(u_i, \hat{u}_{-i}, f) \geq \sup_f J_i(\hat{u}_i, \hat{u}_{-i}, f) - (C_1 + C_2)(1/\sqrt{N} + |x^{(N)}(0) - m_0|).$$

This completes the proof.  $\square$

## 7 Further Generalization to Random Initial States

This section extends the results to a more general model with random initial states. For agent  $\mathcal{A}_i$ , its dynamics are given by

$$dx_i^o(t) = (Ax_i^o(t) + Bu_i(t) + Gx^{o(N)}(t) + f(t))dt + DdW_i(t), \quad 1 \leq i \leq N,$$

where  $x^{o(N)} = (1/N) \sum_{j=1}^N x_j^o$ . The initial states of the agents are given by  $x_i^o(0) = \xi_i$ . As in (4), we define  $J_i(u_i, u_{-i}, f)$  by using  $x_j^o$  in place of  $x_j$ ,  $1 \leq j \leq N$ . Let  $\{\mathcal{F}_t^o\}_{0 \leq t \leq T}$  be the filtration generated by  $\{\xi_i, W_i(t), 1 \leq i \leq N\}$ , and  $L_{\mathcal{F}^o}^2(0, T; \mathbb{R}^k)$  is defined accordingly.

**(H0)** The sequence  $\{\xi_i, i \geq 1\}$  consists of independent random variables which are also independent of the Brownian motions  $\{W_i, i \geq 1\}$ . In addition,  $\lim_{N \rightarrow \infty} (1/N) \sum_{i=1}^N \mathbb{E}\xi_i = m_0$ ,  $\sup_i \mathbb{E}|\xi_i|^2 \leq c_0$  for some constant  $c_0$  independent of  $N$ .

For fixed  $\bar{u}$ , we consider the FBSDE

$$\begin{cases} dx_i^o = (Ax_i^o + BR^{-1}B^T y_i^o + Gm_i^o + \gamma p_i^o)dt + DdW_i, \\ \dot{m}_i^o = (A + G)m_i^o + B\bar{u} + \gamma p_i^o, \\ \dot{p}_i^o = -(A + G)^T p_i^o - (I - \Gamma)^T Q[\mathbb{E}x_i^o - (\Gamma m_i^o + \eta)], \\ dy_i^o = \{-A^T y_i^o + Q[x_i^o - (\Gamma m_i^o + \eta)]\} dt + \zeta_i^o dW_i, \end{cases} \quad (77)$$

where  $x_i^o(0) = \xi_i$ ,  $m_i^o(0) = m_0$ ,  $p_i^o(T) = H\mathbb{E}x_i^o(T)$ , and  $y_i^o(T) = -Hx_i^o(T)$ . Except the random initial state, this FBSDE has the same form as (36).

For the current situation where the filtration is not generated only by the Brownian motions, the proof of Lemma 12 is not applicable. The solution procedure of (P2) as presented in Section 3.5 is only heuristically applied to derive (77). Nevertheless, we can study (77) directly and use it to construct decentralized strategies. We still define  $J_i^{\text{wo}}(u_i, u_{-i}) = \sup_{f \in L^2(0, T; \mathbb{R}^n)} J_i(u_i, u_{-i}, f)$ . The next theorem subsumes Corollary 14 and Theorem 25.

**Theorem 31** *Assume that (H0)-(H2) hold and (42) has a unique solution  $(\mathbf{m}, \mathbf{p}, \mathbf{y})$ . We further take  $\bar{u} = R^{-1}B^T \mathbf{y}$  in (77). Then the two assertions hold.*

- (i) (77) has a unique solution in  $L_{\mathcal{F}^o}^2(0, T; \mathbb{R}^n) \times C^1([0, T]; \mathbb{R}^{2n}) \times L_{\mathcal{F}^o}^2(0, T; \mathbb{R}^{2n})$ .
- (ii) For  $\hat{u}_i = R^{-1}B^T y_i^o$ ,  $1 \leq i \leq N$ , we have

$$J_i^{\text{wo}}(\hat{u}_i, \hat{u}_{-i}) - \varepsilon_N \leq \inf_{u_i \in \mathcal{U}} J_i^{\text{wo}}(u_i, \hat{u}_{-i}) \leq J_i^{\text{wo}}(\hat{u}_i, \hat{u}_{-i}), \quad (78)$$

where  $0 \leq \varepsilon_N = O(1/\sqrt{N} + |(1/N) \sum_{j=1}^N \mathbb{E}\xi_j - m_0|)$ .

*Proof.* (i) Consider (36) by setting

$$\bar{u} = R^{-1}B^T \mathbf{y}, \quad x_i(0) = \mathbb{E}\xi_i. \quad (79)$$

Further construct the ODE by taking expectation in (36):

$$\begin{cases} \dot{\bar{x}}_i = A\bar{x}_i + BR^{-1}B^T\bar{y}_i + G\bar{m}_i + \gamma\bar{p}_i, \\ \dot{\bar{m}}_i = (A + G)\bar{m}_i + B\bar{u} + \gamma\bar{p}_i, \\ \dot{\bar{p}}_i = -(A + G)^T\bar{p}_i - (I - \Gamma)^TQ[\bar{x}_i - (\Gamma\bar{m}_i + \eta)], \\ \dot{\bar{y}}_i = -A^T\bar{y}_i + Q[\bar{x}_i - (\Gamma\bar{m}_i + \eta)], \end{cases} \quad (80)$$

where  $\bar{x}_i(0) = \mathbb{E}\xi_i$ ,  $\bar{m}_i(0) = m_0$ ,  $\bar{p}_i(T) = H\bar{x}_i(T)$ , and  $\bar{y}_i(T) = -H\bar{x}_i(T)$ . Since (36) subject to (79) has a unique solution, (80) has a solution in  $C^1([0, T]; \mathbb{R}^{4n})$ . If (80) has two different solutions, we will be able to construct two different solutions to (36) satisfying (79), a contradiction to Theorem 13. So (80) has a unique solution  $(\bar{x}_i, \bar{m}_i, \bar{p}_i, \bar{y}_i)$ .

Setting  $(m_i^o, p_i^o) = (\bar{m}_i, \bar{p}_i)$  in the first and last equations of (77), we construct the new equations

$$\begin{cases} dx_i^o = (Ax_i^o + BR^{-1}B^T y_i^o + G\bar{m}_i + \gamma\bar{p}_i)dt + DdW_i, \\ dy_i^o = \{-A^T y_i^o + Q[x_i^o - (\Gamma\bar{m}_i + \eta)]\}dt + \zeta_i^o dW_i, \end{cases} \quad (81)$$

where  $x_i^o(0) = \xi_i$  and  $y_i^o(T) = -Hx_i^o(T)$ . Let  $P$  be the solution of the Riccati equation (B.4) and take the transformation  $y_i^o = -Px_i^o + \phi$ . We obtain

$$d\phi = [-(A - BR^{-1}B^T P)^T \phi + P(G\bar{m}_i + \gamma\bar{p}_i) - Q(\Gamma\bar{m}_i + \eta)]dt + (\zeta_i^o + PD)dW_i,$$

where  $\phi(T) = 0$ . We solve  $(\phi, \zeta_i^o) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{2n})$ , and further obtain  $(x_i^o, y_i^o) \in L^2_{\mathcal{F}^o}(0, T; \mathbb{R}^{2n})$ . Subsequently, we can show  $\mathbb{E}x_i^o = \bar{x}_i$ . Hence  $(x_i^o, m_i^o, p_i^o, y_i^o, \zeta_i^o)$  satisfies (77). By taking the variation of the first three equations of (77) and applying an optimal control interpretation as in proving Theorem 13, we can show that  $(x_i^o, m_i^o, p_i^o, y_i^o, \zeta_i^o)$  is the unique solution.

(ii) By slightly modifying the proofs of Theorem 18 and the associated lemmas, we can show

$$\sup_{0 \leq t \leq T} \mathbb{E}|\hat{u}^{(N)} - \bar{u}^*|^2 = O(1/N) + O(|\mathbb{E}x^{(N)}(0) - m_0|^2).$$

Next, we adapt the proofs of Lemmas 26-30 taking into account the random initial states satisfying (H0). This gives the desired estimate for  $\varepsilon_N$ .  $\square$

## 8 Conclusion

This paper introduces a class of mean field LQG games with drift uncertainty. By using the idea of robust optimization, the local strategy is designed by minimizing the worst case cost. When the decentralized strategies are implemented in a finite population, their performance is characterized as a robust  $\varepsilon$ -Nash equilibrium.

In this paper we only deal with drift uncertainty. If the Brownian motions are also subject to an uncertain coefficient process to model volatility uncertainty [38], the resulting optimal control problems will give a set of more complicated FBSDE. It is also of potential interest to address model uncertainty of the mean field game in a different setup by considering measure uncertainty [16, 36, 48] in the robust optimization problem. This will necessitate the use of different techniques for analysis.

## Appendix A

For proving Lemma 9, we give another lemma first. Consider an auxiliary optimal control problem with dynamics

$$\begin{cases} \dot{z}_i = Az_i + Bv_i + Gz + \gamma q, \\ \dot{z} = (A + G)z + \gamma q, \\ \dot{q} = -(A + G)^T q - (I - \Gamma)^T Q(\mathbb{E}z_i - \Gamma z), \end{cases} \quad (\text{A.1})$$

where  $z_i(0) = z(0) = 0$ ,  $q(T) = H\mathbb{E}z_i(T)$  and  $v_i \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$ . Following the argument in the proof of Lemma 8, under (H1) we can show the existence and uniqueness of a solution to (A.1). The optimal control problem is

$$(\text{P2b}) \quad \text{minimize} \quad \bar{J}_i^b(v_i) = \mathbb{E} \int_0^T \{ |z_i - \Gamma z|_Q^2 + v_i^T R v_i - \gamma |q(t)|^2 \} dt + \mathbb{E} |z_i(T)|_H^2.$$

Similarly, we may define positive definiteness of  $\bar{J}_i^b$  as in Section 3.

**Lemma A.1**  $\bar{J}_i^a$  is positive semi-definite (resp., positive definite) if and only if  $\bar{J}_i^b$  is positive semi-definite (resp., positive definite).

*Proof.* It suffices to show the “only if” part.

Suppose that  $\bar{J}_i^a$  is positive semi-definite. Consider any control  $v_i \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$  for  $\bar{J}_i^b$ , and this gives a unique solution  $(z_i, z, q)$ . We take expectation in (A.1) to obtain

$$\begin{cases} \dot{\bar{z}}_i = A\bar{z}_i + B\bar{v}_i + Gz + \gamma q, \\ \dot{z} = (A + G)z + \gamma q, \\ \dot{q} = -(A + G)^T q - (I - \Gamma)^T Q(\bar{z}_i - \Gamma z), \end{cases}$$

where  $\bar{z}_i = \mathbb{E}z_i$  and  $\bar{v}_i = \mathbb{E}v_i$ .

It follows that

$$\begin{aligned} \bar{J}_i^b(v_i) &= \bar{J}_i^a(\bar{v}_i) + \mathbb{E} \int_0^T [ |z_i - \mathbb{E}z_i|_Q^2 + |v_i - \mathbb{E}v_i|_R^2 ] dt + \mathbb{E} |z_i(T) - \mathbb{E}z_i(T)|_H^2 \\ &\geq \bar{J}_i^a(\bar{v}_i) \geq 0. \end{aligned}$$

On the other hand,  $\bar{J}_i^a(0) = 0$ . This shows that  $\bar{J}_i^b$  is positive semi-definite. The above reasoning is also valid for the positive definite case. This proves the “only if” part.  $\square$

*Proof of Lemma 9.* Let  $(x_i, m_i, p_i)$  and  $(x'_i, m'_i, p'_i)$  be two state processes in (P2) corresponding to the controls  $u_i$  and  $u'_i$ , respectively. Assume  $\lambda_1 \in [0, 1]$  and  $\lambda_1 + \lambda_2 = 1$ . We have

$$\begin{aligned} &\lambda_1 \bar{J}_i(u_i) + \lambda_2 \bar{J}_i(u'_i) - \bar{J}_i(\lambda_1 u_i + \lambda_2 u'_i) \\ &= \lambda_1 \lambda_2 \mathbb{E} \int_0^T \{ |x_i - x'_i - \Gamma(m_i - m'_i)|_Q^2 + |u_i - u'_i|_R^2 - \gamma |p_i(t) - p'_i(t)|^2 \} dt \\ &\quad + \lambda_1 \lambda_2 \mathbb{E} |x_i(T) - x'_i(T)|_H^2. \end{aligned}$$

Denote  $z_i = x_i - x'_i$ ,  $z = m_i - m'_i$ ,  $q = p_i - p'_i$  and  $v_i = u_i - u'_i$ . It is obvious

$$\lambda_1 \bar{J}_i(u_i) + \lambda_2 \bar{J}_i(u'_i) - \bar{J}_i(\lambda_1 u_i + \lambda_2 u'_i) = \lambda_1 \lambda_2 \bar{J}_i^b(v_i).$$

Recalling Lemma A.1, this completes the proof.  $\square$

## Appendix B

We introduce the FBSDE

$$\begin{cases} dx_i = (Ax_i + BR^{-1}B^T y_i + Gm_i + \gamma p_i)dt + DdW_i, \\ \dot{m}_i = (A + G)m_i + BR^{-1}B^T \mathbb{E}y_i + \gamma p_i, \\ \dot{p}_i = -(A + G)^T p_i - (I - \Gamma)^T Q[m_i - (\Gamma m_i + \eta)], \\ dy_i = \{-A^T y_i + Q[x_i - (\Gamma m_i + \eta)]\} dt + \zeta_i dW_i, \end{cases} \quad (\text{B.1})$$

where  $x_i(0) = m_i(0) = m_0$ ,  $p_i(T) = Hm_i(T)$ , and  $y_i(T) = -Hx_i(T)$ . This FBSDE is slightly different from (42) by the third equation and the condition on  $p_i(T)$  and will be more convenient for analysis.

The next lemma shows that the two equation systems (43) and (B.1) are equivalent. The proof is straightforward since  $\mathbb{E}x_i$  and  $m_i$  satisfy the same ODE with the same initial condition.

**Lemma B.1** *If  $(x_i, m_i, p_i, y_i, \zeta_i) \in S[0, T]$  satisfies one of (43) and (B.1), it also satisfies the other.  $\square$*

Consider the ODE system

$$\begin{cases} \dot{\bar{x}}_i = A\bar{x}_i + BR^{-1}B^T \bar{y}_i + G\bar{m}_i + \gamma \bar{p}_i, \\ \dot{\bar{m}}_i = (A + G)\bar{m}_i + BR^{-1}B^T \bar{y}_i + \gamma \bar{p}_i, \\ \dot{\bar{p}}_i = -(A + G)^T \bar{p}_i - (I - \Gamma)^T Q[\bar{m}_i - (\Gamma \bar{m}_i + \eta)], \\ \dot{\bar{y}}_i = -A^T \bar{y}_i + Q[\bar{x}_i - (\Gamma \bar{m}_i + \eta)], \end{cases} \quad (\text{B.2})$$

where  $\bar{x}_i(0) = \bar{m}_i(0) = m_0$ ,  $\bar{p}_i(T) = H\bar{m}_i(T)$  and  $\bar{y}_i(T) = -H\bar{x}_i(T)$ .

**Lemma B.2** *The two statements are equivalent:*

- (i) *The FBSDE (B.1) has a unique solution in  $S[0, T]$ .*
- (ii) *The ODE (B.2) has a unique solution in  $C^1([0, T]; \mathbb{R}^{4n})$ .*

*Proof.* Step 1. Suppose that (ii) holds and let the unique solution be denoted by  $(\bar{x}_i, \bar{m}_i, \bar{p}_i, \bar{y}_i)$ .

Take  $(m_i, p_i) = (\bar{m}_i, \bar{p}_i)$  on the right hand side of the first and last equations of (B.1) to write

$$\begin{cases} dx_i = (Ax_i + BR^{-1}B^T y_i + G\bar{m}_i + \gamma \bar{p}_i)dt + DdW_i, \\ dy_i = \{-A^T y_i + Q[x_i - (\Gamma \bar{m}_i + \eta)]\} dt + \zeta_i dW_i, \end{cases} \quad (\text{B.3})$$

where  $y_i(T) = -Hx_i(T)$ . Denote the Riccati equation

$$\dot{P} + A^T P + PA - PBR^{-1}B^T P + Q = 0, \quad P(T) = H, \quad (\text{B.4})$$

which has a unique solution on  $[0, T]$ . Setting  $y_i = -Px_i + \phi$  in (B.3), we obtain two decoupled equations for  $(x_i, \phi)$  which is uniquely solved. This further gives a unique solution  $(x_i, y_i, \zeta_i) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{3n})$  for (B.3). Taking expectation on both sides of (B.3) yields

$$\begin{cases} \frac{d}{dt} \mathbb{E}x_i = A\mathbb{E}x_i + BR^{-1}B^T \mathbb{E}y_i + G\bar{m}_i + \gamma \bar{p}_i, \\ \frac{d}{dt} \mathbb{E}y_i = -A^T \mathbb{E}y_i + Q[\mathbb{E}x_i - (\Gamma \bar{m}_i + \eta)], \end{cases} \quad (\text{B.5})$$

where  $\mathbb{E}y_i(T) = -H\mathbb{E}x_i(T)$ . By combining (B.5) with the first and fourth equations of (B.2), it is easy to show  $\mathbb{E}x_i = \bar{x}_i$  and  $\mathbb{E}y_i = \bar{y}_i$  for all  $t \in [0, T]$ . This implies

$$\begin{aligned} \dot{\bar{m}}_i &= (A + G)\bar{m}_i + BR^{-1}B^T \bar{y}_i + \gamma \bar{p}_i \\ &= (A + G)\bar{m}_i + BR^{-1}B^T \mathbb{E}y_i + \gamma \bar{p}_i. \end{aligned}$$

The third equation of (B.1) is clearly satisfied by  $(\bar{m}_i, \bar{p}_i)$ . Therefore,  $(x_i, m_i, p_i, y_i, \zeta_i) := (x_i, \bar{m}_i, \bar{p}_i, y_i, \zeta_i)$  satisfies (B.1).

We continue to show that  $(x_i, m_i, p_i, y_i, \zeta_i)$  above is the unique solution of (B.1). Suppose that  $(x'_i, m'_i, p'_i, y'_i, \zeta'_i)$  is another solution of (B.1). It is clear that  $(\mathbb{E}x'_i, m'_i, p'_i, \mathbb{E}y'_i)$  is a solution of (B.2). Since (B.2) has a unique solution  $(\bar{x}_i, \bar{m}_i, \bar{p}_i, \bar{y}_i)$ , we have  $(m'_i, p'_i) = (\bar{m}_i, \bar{p}_i)$ . By using the first and fourth equations of (B.1), we derive the equations satisfied by  $(x'_i - x_i, y'_i - y_i)$  and further infer  $(x'_i, y'_i) = (x_i, y_i)$ . We conclude that (i) holds.

Step 2. Suppose that (i) holds with the unique solution denoted by  $(x_i, m_i, p_i, y_i, \zeta_i)$ . It is obvious that  $(\bar{x}_i, \bar{m}_i, \bar{p}_i, \bar{y}_i) := (\mathbb{E}x_i, m_i, p_i, \mathbb{E}y_i)$  is a solution of (B.2). Suppose that  $(\bar{x}'_i, \bar{m}'_i, \bar{p}'_i, \bar{y}'_i) \neq (\bar{x}_i, \bar{m}_i, \bar{p}_i, \bar{y}_i)$  is another solution of (B.2). Then  $(x_i, m_i, p_i, y_i, \zeta_i) + (\bar{x}'_i - \bar{x}_i, \bar{m}'_i - \bar{m}_i, \bar{p}'_i - \bar{p}_i, \bar{y}'_i - \bar{y}_i, 0)$  is also a solution of (B.1), a contradiction to (i). So (B.2) has a unique solution.  $\square$

**Lemma B.3** (i) If  $(\bar{x}_i, \bar{m}_i, \bar{p}_i, \bar{y}_i)$  is a solution of (B.2),  $(\mathbf{m}, \mathbf{p}, \mathbf{y}) := (\bar{m}_i, \bar{p}_i, \bar{y}_i)$  satisfies (42).

(ii) If  $(\mathbf{m}, \mathbf{p}, \mathbf{y})$  is a solution of (42), there exists  $\bar{x}_i$  such that  $(\bar{x}_i, \bar{m}_i, \bar{p}_i, \bar{y}_i) := (\bar{x}_i, \mathbf{m}, \mathbf{p}, \mathbf{y})$  satisfies (B.2).

(iii) The ODE (B.2) has a unique solution if and only if (42) has a unique solution.

*Proof.* (i) If  $(\bar{x}_i, \bar{m}_i, \bar{p}_i, \bar{y}_i)$  is a solution of (B.2),  $\bar{x}_i = \bar{m}_i$  and therefore  $\bar{y}_i(T) = -H\bar{x}_i(T) = -H\bar{m}_i(T)$ . So  $(\mathbf{m}, \mathbf{p}, \mathbf{y})$  defined above satisfies (42).

(ii) If  $(\mathbf{m}, \mathbf{p}, \mathbf{y})$  is a solution of (42), we set  $(\bar{m}_i, \bar{p}_i, \bar{y}_i) = (\mathbf{m}, \mathbf{p}, \mathbf{y})$  and define  $\bar{x}_i$  by the ODE

$$\dot{\bar{x}}_i = A\bar{x}_i + BR^{-1}B^T\bar{y}_i + G\bar{m}_i + \gamma\bar{p}_i,$$

where  $\bar{x}_i(0) = m_0$ . It can be checked that  $\bar{m}_i = \bar{x}_i$ , which gives  $\bar{y}_i(T) = -H\bar{m}_i(T) = -H\bar{x}_i(T)$ . Hence,  $(\bar{x}_i, \bar{m}_i, \bar{p}_i, \bar{y}_i)$  is a solution to (B.2).

(iii) Assume that (42) has a unique solution. Let  $(\bar{x}_i, \bar{m}_i, \bar{p}_i, \bar{y}_i)$  and  $(\bar{x}'_i, \bar{m}'_i, \bar{p}'_i, \bar{y}'_i)$  be two solutions of (B.2). By (i),  $(\bar{m}_i, \bar{p}_i, \bar{y}_i)$  and  $(\bar{m}'_i, \bar{p}'_i, \bar{y}'_i)$  are two solutions of (42) and so must be equal, which further implies  $\bar{x}_i = \bar{x}'_i$  by the first equation of (B.2). This shows that (B.2) has a unique solution.

Next assume that (B.2) has a unique solution. Let  $(\mathbf{m}, \mathbf{p}, \mathbf{y})$  and  $(\mathbf{m}', \mathbf{p}', \mathbf{y}')$  be two solutions of (42). By (ii), we must have  $(\mathbf{m}, \mathbf{p}, \mathbf{y}) = (\mathbf{m}', \mathbf{p}', \mathbf{y}')$ . Therefore, (42) has a unique solution.  $\square$

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